Hand-out 6
Calculus II
Handy Dandy Guide to Series Part III

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Spring 2012

A **power series at** $\mathbf{x = c}$ is a series such that it is of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n \qquad \forall \quad n \in \mathbb{N}^* \quad a_n \in \mathbb{R}$$

.

Let $D \subseteq \mathbb{R} \quad \wedge \quad [a,b] \subseteq D$.

Let $f : D \longrightarrow \mathbb{R}$ be a well defined function such that $f^{(n)}(x)$ exists $\quad \forall \quad x \in (a,b)$.

Let $\quad c \in (a,b)$.

The **Taylor Series for** $f(\mathbf{x})$ **at** $\mathbf{x = c}$ is a power series such that we will denote it as $T(x)$ where

$$f(x) = T(x) = f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x)\,(x-c)^n}{n!} \quad \forall x \in K$$

where K is the set of all points in the interval of convergence for $T(x)$. So, obviously $c \in K$.

The Maclaurin series is the case $a = 0$ for a Taylor series of the function $f(x)$.
There are some Maclaurin series of import that we need to recall or produce quickly for applications and approximations, which are as follows:

(1)

$$e^x = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \quad \forall x \in \mathbb{R}$$

(2)

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \forall x \in \mathbb{R}$$

(3)

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \quad \forall x \in [-1,1]$$

(4)

$$\ln(1+x) = \sum_{n=}^{\infty} \frac{(-1)^{(n+1)} x^n}{n} \quad \forall x \in (-1,1]$$

(5)

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \forall x \in \mathbb{R}$$

(6)

$$\frac{1}{1-x} = 1 + \sum_{n=1}^{\infty} x^n \quad \forall x \in (-1,1)$$

If one is smart then they DO NOT have to memorise these others for they are easily derived:

(1)
$$\cos x = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \forall x \in \mathbb{R}$$

(2)
$$\cosh x = 1 + \sum_{n=1}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \forall x \in \mathbb{R}$$

(3)
$$\frac{1}{1+x^2} = 1 + \sum_{n=1}^{\infty} (-1)^n \cdot x^{2n} \quad \forall x \in (-1,1)$$

This begs the question, why are they easily derived?
Well, for example consider

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \quad \forall x \in [-1,1]$$

$$f(x) = \arctan(x) \Rightarrow f^{(1)}(x) = \frac{1}{1+x^2}$$