Handout $2\frac{1}{2}$
The Lemmas and Theorems of Use to Us

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Let our universe be $\mathbb{R} \times \mathbb{R}$ which is the Cartesian plane. Let $D \subseteq \mathbb{R}$

Lemma 0: Let $f : D \longrightarrow \mathbb{R}$ and $g : D \longrightarrow \mathbb{R}$ be well defined functions.
Let a, b, and c be real numbers such that $(a, b) \cup (b, c) \subseteq D$.
Let $h : D \longrightarrow \mathbb{R}$ be the well defined function such that $h(x) = \dfrac{f(x)}{g(x)}$.
Suppose $\lim_{x\to b} f(x) = 0$ whilst $\lim_{x\to b} g(x) = 0$. This tells us **nothing** about $\lim_{x\to b} h(x)$.

Theorem 0: Let $f : D \longrightarrow \mathbb{R}$ and $g : D \longrightarrow \mathbb{R}$ be well defined functions.
Let a, b, and c be real numbers such that $(a, b) \cup (b, c) \subseteq D$.
Let $h : D \longrightarrow \mathbb{R}$ be the well defined function such that $h(x) = \dfrac{f(x)}{g(x)}$
Suppose $\lim_{x\to b} f(x) = p$ where $p \in \mathbb{R} \wedge p \neq 0$ whilst $\lim_{x\to b} g(x) = 0$,
then it is the case that $\lim_{x\to b} h(x)$ does not exist.

Theorem 1: Let $f : D \longrightarrow \mathbb{R}$ and $g : D \longrightarrow \mathbb{R}$ be well defined functions.
Let a be a real number such that $(a, \infty) \subseteq D$.
Let $f(x) = p$ where $p \in \mathbb{R}$. Let $h : D \longrightarrow \mathbb{R}$ be the well defined function such that $h(x) = \dfrac{f(x)}{g(x)}$
Suppose $\lim_{x\to\infty} g(x)$ blows up (∞ sense) or blows down ($-\infty$ sense), so the numerator is constant and the denominator blows up (or down) thus it is the case that $\lim_{x\to b} h(x) = 0$.

Theorem 1A: Let $f : D \longrightarrow \mathbb{R}$ and $g : D \longrightarrow \mathbb{R}$ be well defined functions.
Let a be a real number such that $(a, \infty) \subseteq D$.
Let $h : D \longrightarrow \mathbb{R}$ be the well defined function such that $h(x) = \dfrac{f(x)}{g(x)}$
Suppose $\lim_{x\to\infty} f(x) = q$ where $q \in \mathbb{R}$ whilst $\lim_{x\to\infty} g(x)$ blows up (∞ sense) or blows down ($-\infty$ sense),
then it is the case that $\lim_{x\to b} h(x) = 0$.

Theorem 1B: Let $f : D \longrightarrow \mathbb{R}$ and $g : D \longrightarrow \mathbb{R}$ be well defined functions.
Let a be a real number such that $(a, \infty) \subseteq D$.
Let $h : D \longrightarrow \mathbb{R}$ be the well defined function such that $h(x) = \dfrac{f(x)}{g(x)}$
Suppose there is a real number b such that $|f(x)| \leq b$ for $x \in (a, \infty)$ (meaning f is bounded)
and $\lim_{x\to\infty} g(x)$ blows up (∞ sense) or blows down ($-\infty$ sense), then it is the case that $\lim_{x\to b} h(x) = 0$.

Lemma 0A: Let $f : D \longrightarrow \mathbb{R}$ and $g : D \longrightarrow \mathbb{R}$ be well defined functions.
Let a be a real number such that $(a, \infty) \subseteq D$.
Let $h : D \longrightarrow \mathbb{R}$ be the well defined function such that $h(x) = \dfrac{f(x)}{g(x)}$.
Suppose $\lim\limits_{x\to\infty} f(x) = 0$ whilst $\lim\limits_{x\to\infty} g(x) = 0$. This tells us **nothing** about $\lim\limits_{x\to\infty} h(x)$.

Lemma 0B: Let $f : D \longrightarrow \mathbb{R}$ and $g : D \longrightarrow \mathbb{R}$ be well defined functions.
Let a, b, and c be real numbers such that $(a, b) \cup (b, c) \subseteq D$.
Let $h : D \longrightarrow \mathbb{R}$ be the well defined function such that $h(x) = \dfrac{f(x)}{g(x)}$.
Suppose $\lim\limits_{x\to b} f(x)$ blows up (∞ sense) or blows down ($-\infty$ sense),
whilst $\lim\limits_{x\to b} g(x)$ blows up (∞ sense) or blows down ($-\infty$ sense).
This tells us **nothing** about $\lim\limits_{x\to b} h(x)$.

Lemma 1: Let $f : D \longrightarrow \mathbb{R}$ and $g : D \longrightarrow \mathbb{R}$ be well defined functions.
Let a be a real number such that $(a, \infty) \subseteq D$.
Let $h : D \longrightarrow \mathbb{R}$ be the well defined function such that $h(x) = \dfrac{f(x)}{g(x)}$.
Suppose $\lim\limits_{x\to\infty} f(x)$ blows up (∞ sense) or blows down ($-\infty$ sense), whilst $\lim\limits_{x\to\infty} g(x)$ blows up (∞ sense) or blows down ($-\infty$ sense). This tells us **nothing** about $\lim\limits_{x\to\infty} h(x)$.