

Handout I

The Concept of Limit of a Function at $x = a$

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Let our universe be $\mathbb{R} \times \mathbb{R}$ which is the Cartesian plane. Let $D \subseteq \mathbb{R}$

§1 The Concept of Limit of a Function at $x = a$

Definition 1.1A: Let $f : D \rightarrow \mathbb{R}$ be a well defined function.

Let $x = a$ (or simply a) be a real number such that there are some positive real numbers b and c so that $(a - b, a) \subseteq D$ and $(a + c, a) \subseteq D$.

Let $y = l$ (or we just say the real number l).

We say the limit of $f(x)$ at $x = a$ is $y = l$ if and only if

for each lines $y = m$ and $y = p$ where $m < l < p$

there exist lines $x = h$ and $x = k$ where $h < a < k$

so that for each $x \in D$ that is also in (h, a) or (a, k)

it is the case that $f(x) \in (m, p)$.

The notation for this is $\lim_{x \rightarrow a} f(x) = l$, $\lim_{x \rightarrow a} (f(x)) = l$, or $f(x) \xrightarrow{x \rightarrow a} l$.

Definition 1.1B: Let $f : D \rightarrow \mathbb{R}$ be a well defined function.

Let $x = a$ (or simply a) be a real number such that there is some positive real number β so that $(a - \beta, a) \cup (a, a + \beta) \subseteq D$.

Let $y = L$ (or we just say the real number L).

We say the limit of $f(x)$ at $x = a$ is $y = L$ if and only if

$\forall \epsilon > 0 \quad \exists \delta > 0 \ni$

$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$.

Definition 1.1C: Let $f : D \rightarrow \mathbb{R}$ be a well defined function.

Let $x = a$ (or simply a) be a real number such that there is some positive real number β so that $(a - \beta, a) \cup (a, a + \beta) \subseteq D$.

Let $y = L$ (or we just say the real number L).

We say the limit of $f(x)$ at $x = a$ is $y = L$ if and only if

$\forall \epsilon > 0 \quad \exists \delta > 0 \ni$

$|f(x) - L| < \epsilon$ whenever it is true that $0 < |x - a| < \delta$.

Definition 1.2: Let $f : D \rightarrow \mathbb{R}$ be a well defined function.

Let $x = a$ (or simply a) be a real number such that there is some positive real number β so that $(a - \beta, a) \cup (a, a + \beta) \subseteq D$.

We say the limit of $f(x)$ at $x = a$ *does not exist* (**DNE** for short) if and only if

$\exists \epsilon > 0$ such that $\forall \delta > 0$ it is the case that $0 < |x - a| < \delta \nRightarrow |f(x) - L| < \epsilon$ which means there is some $x_1 \neq a$ where $x_1 \in (a - \delta, a + \delta)$ so that forces $0 < |x_1 - a| < \delta$ but it is the case that $|f(x_1) - L| \geq \epsilon$

For all of the above the notation for limit when it exists and is l is:

$$\lim_{x \rightarrow a} f(x) = l$$

and when it does not exist the notation and shorthand should be written as:

$$\lim_{x \rightarrow a} f(x) \quad DNE$$

§2 The Naïve Concept of Limit of a Function at $x = a$

'Conceptual' Definition 2.1: Let $f : D \rightarrow \mathbb{R}$ be a well defined function. Let $x = a$ (or simply a) be a real number such that there are some positive real numbers b and c so that $(a - b, a) \subseteq D$ and $(a + c, a) \subseteq D$. Let $y = l$.

Let the sequence, $\{x_n\}_{n=1}^{\infty} \subseteq (a - b, a)$ be 'any old sequence "crunching" toward a from the left' and $\{x_n\}_{n=1}^{\infty} \rightarrow a$

We say the 'left limit' of $f(x)$ for $x \rightarrow a^-$ is $y = l$ if and only if $\{f(x_n)\}_{n=1}^{\infty} \rightarrow l$

The notation for this is $\lim_{x \rightarrow a^-} f(x) = l$

'Conceptual' Definition 2.2: Let $f : D \rightarrow \mathbb{R}$ be a well defined function. Let $x = a$ (or simply a) be a real number such that there are some positive real numbers b and c so that $(a - b, a) \subseteq D$ and $(a + c, a) \subseteq D$.

Let $y = l$.

Let the sequence, $\{x_n\}_{n=1}^{\infty} \subseteq (a, a + c)$ be 'any old sequence "crunching" toward a from the right' and $\{x_n\}_{n=1}^{\infty} \rightarrow a$

We say the 'right limit' of $f(x)$ for $x \rightarrow a^+$ is $y = l$ if and only if $\{f(x_n)\}_{n=1}^{\infty} \rightarrow l$

The notation for this is $\lim_{x \rightarrow a^+} f(x) = l$

'Conceptual' Definition 2.3: Let $f : D \rightarrow \mathbb{R}$ be a well defined function. Let $x = a$ (or simply a) be a real number such that there are some positive real numbers b and c so that $(a - b, a) \subseteq D$ and $(a + c, a) \subseteq D$.

Let $y = l$.

We say the limit of $f(x)$ is l if and only if

$$\lim_{x \rightarrow a^-} f(x) = l \text{ and}$$

$$\lim_{x \rightarrow a^+} f(x) = l$$

§3 Examples of Naïve Concept of Limit of a Function at $x = a$

The idea is that there is the sequence, $\{x_n\}_{n=1}^{\infty} \subseteq (a - b, a)$ is 'any old sequence "crunching" toward a from the left' and $\{x_n\}_{n=1}^{\infty} \rightarrow a$ whilst there is the sequence, $\{x_m\}_{m=1}^{\infty} \subseteq (a, a + c)$ is 'any old sequence "crunching" toward a from the right' and $a \leftarrow \{x_m\}_{m=1}^{\infty}$ and both $\{f(x_n)\}_{n=1}^{\infty} \rightarrow l$ and $\{f(x_m)\}_{m=1}^{\infty} \rightarrow l$

Example 3.1: Let $f : D \rightarrow \mathbb{R}$ be such that $D = (0, \infty)$ and $f(x) = \sqrt[3]{x + 2}$.

We claim $\lim_{x \rightarrow 25} f(x) = 3$.

We show this by 'crunching numbers, first, toward 25 from the left and notice the values:

$$24 \quad f(24) = \sqrt[3]{26} \approx 2.9624960684$$

$$24.9 \quad f(24.9) = \sqrt[3]{26.9} \approx 2.99629171439$$

$$24.99 \quad f(24.99) = \sqrt[3]{26.99} \approx 2.9996295838954$$

$$\begin{array}{ccc}
 24.999 & f(24.999) = \sqrt[3]{26.999} \approx 2.9999629625057 & \\
 \downarrow & & \downarrow \\
 25^- & & 3
 \end{array}$$

then from the right:

$$\begin{array}{ccc}
 26 & f(24) = \sqrt[3]{28} \approx 3.0365889718 & \\
 25.1 & f(25.1) = \sqrt[3]{27.1} \approx 3.003699 & \\
 25.01 & f(25.01) = \sqrt[3]{27.01} \approx 3.00037 & \\
 25.001 & f(25.001) = \sqrt[3]{27.001} \approx 3.000037037 & \\
 \downarrow & & \downarrow \\
 25^+ & & 3
 \end{array}$$

So, we say

$$\lim_{x \rightarrow 25} \sqrt[3]{x+2} = 3$$

Example 3.2: Let $f : D \rightarrow \mathbb{R}$ be such that $D = (-\infty, 1) \cup (1, 2) \cup (2, \infty)$ and $f(x) = \frac{x^2-1}{x^2-3x+2}$. We claim $\lim_{x \rightarrow 1} f(x) = -2$.

Notice $1 \notin D$ but we can still try to find a limit for $f(x)$ at $x = 1$.

We show this by 'crunching numbers, first, toward 1 from the left and notice the values:

$$\begin{array}{ccc}
 0 & f(0) = -\frac{1}{2} & \\
 \frac{9}{10} & f\left(\frac{9}{10}\right) = -\frac{19}{11} & \\
 \frac{99}{100} & f\left(\frac{99}{100}\right) = -\frac{199}{101} & \\
 \downarrow & \downarrow & \\
 1^- & -2 &
 \end{array}$$

$$\begin{array}{ccc}
 0 & f(2) \text{ does not exist} & \\
 \frac{11}{10} & f\left(\frac{11}{10}\right) = -\frac{21}{11} & \\
 \frac{101}{100} & f\left(\frac{101}{100}\right) = -\frac{201}{101} & \\
 \downarrow & \downarrow & \\
 1^+ & -2 &
 \end{array}$$

So, we say

$$\lim_{x \rightarrow 1} \frac{x^2-1}{x^2-3x+2} = -2$$

Example 3.3: Consider $k : \mathbb{R} \rightarrow \mathbb{R}$ where

$$k(x) = \begin{cases} 3x+15, & x < -2 \\ 3, & x = -2 \\ 9, & -2 < x < 1 \\ \frac{4}{(x+2)^2}, & 1 \leq x < 3 \\ (x+1)^3, & x \geq 3 \end{cases}$$

When we consider $\lim_{x \rightarrow 3^-} k(x)$ we get $\frac{4}{25}$

When we consider $\lim_{x \rightarrow 3^+} k(x)$ we get 64

Since $\frac{4}{25} \neq 64 \lim_{x \rightarrow 3} k(x)$ **DNE**.

When we consider $\lim_{x \rightarrow -2^-} k(x)$ we get 9

When we consider $\lim_{x \rightarrow -2^+} k(x)$ we get 9

Since $9 = 9 \lim_{x \rightarrow 3} k(x) = 9$. Please notice $f(-2)$ is not 9. Wonderful! Notice that limits at $x = a$ do not depend on function value at $x = a$.

Example 3.4: Let $D = [0, \infty)$. Consider $j : D \rightarrow \mathbb{R}$ where

$$j(x) = \begin{cases} 3x - 44, & 0 \leq x < 4 \\ 4 - 9x, & 4 < x < 10 \\ \sin x, & x \geq 10 \end{cases}$$

When we try to consider $\lim_{x \rightarrow 0^-} j(x)$ we get nothing - it doesn't exist since there are no values of D in $(-\infty, 0)$. End of story.

$\lim_{x \rightarrow 3} j(x)$ **DNE**.

When we consider $\lim_{x \rightarrow 4^-} j(x)$ we get -32

When we consider $\lim_{x \rightarrow 4^+} j(x)$ we get -32

Since $-32 = -32$

$\lim_{x \rightarrow 4} j(x) = -32!$

When we consider $\lim_{x \rightarrow 10^-} j(x)$ we get -86

When we consider $\lim_{x \rightarrow 10^+} j(x)$ we get $\sin 10$

Since $-86 \neq \sin 10$ the limit does not exist for $j(x)$ at $x = 10$.

§4 Naïve Concept of 'Connected' Curves In The Plane

'Definition' 4.1: Consider our universe $\mathbb{R} \times \mathbb{R}$ which is the Cartesian plane.

Let \mathcal{C} be a curve in the plane. We say **the curve \mathcal{C} is planarly connected (or just connected for short)** if and only if for each two points, p and q , of \mathcal{C} there is an arc 'connecting' them that is a subset of \mathcal{C} . In a vernacular sense, we say planarly connected curve \mathcal{C} has no jumps, no holes, no vertical asymptotes, etc. which 'cut up' the curve.

Question 4.1: Example 3.1 is planarly connected. Why?

Question 4.2: Example 3.2 is not planarly connected. Why?

Question 4.3: Example 3.3 is not planarly connected. Why?

Question 4.4: Example 3.4 is not planarly connected. Why?

Question 4.5: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that $f(x) = |x|$. Is f planarly connected or not? Why?

Question 4.6: Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be such that $g(x) = \sin(x)$. Is f planarly connected or not? Why?

Question 4.7: Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be such that $h(x) = 3 \cdot x + 5$ when $x < 3$ and $h(x) = x^2$ when $x \geq 3$. Is h planarly connected or not? Why?

Question 4.8: Let $k : \mathbb{R} \rightarrow \mathbb{R}$ be such that $k(x) = 3 \cdot x + 5$ when $x < 3$ and $h(x) = 23 - x^2$ when $x \geq 3$. Is k planarly connected or not? Why?

Question 4.9: Let $p : (-\infty, 3) \cup (3, \infty) \rightarrow \mathbb{R}$ be such that

$$p(x) = \frac{3x^2 - 8x - 3}{x - 3}$$

. Is p planarly connected or not? Why?

§5 Naïve Concept of Limit of a Function as x 'blows up' or 'blows down.'

'Conceptual' Definition 5.1: Let $f : D \rightarrow \mathbb{R}$ be a well defined function. Let a be a real number such that $[a, \infty) \subseteq D$.

Let $y = l$ (or we just say the real number l).

Let the sequence, $\{x_n\}_{n=1}^{\infty}$ be 'any old sequence "growing endlessly" (unbounded) blowing up and going farther and farther to the right in the plane a . We say the 'limit as x blows up' of $f(x)$ is $y = l$ if and only if $\{f(x_n)\}_{n=1}^{\infty} \rightarrow l$

The notation for this is $\lim_{x \rightarrow \infty} f(x) = l$

'Conceptual' Definition 5.2: Let $f : D \rightarrow \mathbb{R}$ be a well defined function. Let b be a real number such that $(-\infty, b) \subseteq D$.

Let $y = l$ (or we just say the real number l).

Let the sequence, $\{x_n\}_{n=1}^{\infty}$ be 'any old sequence "shrinking endlessly" (unbounded) blowing down and going farther and farther to the left in the plane a . We say the 'limit as x blows down' of $f(x)$ is $y = l$ if and only if $\{f(x_n)\}_{n=1}^{\infty} \rightarrow l$

The notation for this is $\lim_{x \rightarrow -\infty} f(x) = l$

Question 5.1: Does there exist a well defined function in the plane such that it has neither a blow up nor blow down limit? Explain.

Question 5.2: Does there exist a well defined function in the plane such that it has both a blow up and a blow down limit? Explain.

Question 5.3: Does there exist a well defined function in the plane such that it has a blow up limit but not a blow down limit? Explain.

Question 5.4: Does there exist a well defined function in the plane such that it has a blow down limit but not a blow up limit? Explain.

Question 5.5: Does there exist a well defined function in the plane such that it has a blow up and a blow down limit but those two real numbers are **different**? Explain.

Theorem 5.6: Let $D \subseteq \mathbb{R}$, a be a real number such that $[a, \infty) \subseteq D$. Let $f : D \rightarrow \mathbb{R}$ be a well defined function. Let b be a real number and c be a positive real number. Let

$$f(x) = \frac{b}{x^c}$$

It is the case that $\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow \infty} f(x) = 0$.