Handout II

The Concept of Continuity of a Function at the Point$(a, f(a))$

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Let our universe be $\mathbb{R} \times \mathbb{R}$ which is the Cartesian plane. Let $D \subseteq \mathbb{R}$

You will notice that continuity of $f(x)$ at $(a, f(a))$ is just a 'tad' more developed than limit and is built upon that concept.

Definition 2.1A: Let $f : D \longrightarrow \mathbb{R}$ be a well defined function.
Let $x = a$ (or simply a) be a real number such that there is some positive real number b so that $(a - b, a + b) \subseteq D$.
Let $y = f(a)$ we say the function f is continuous at $(a, f(a))$ if and only if
for each lines $y = m$ and $y = p$ where $m < f(a) < p$
there exist lines $x = h$ and $x = k$ where $h < a < k$
so that for each $x \in D$ that is also in (h, k)
it is the case that $f(x) \in (m, p)$.

Definition 2.1C: Let $f : D \longrightarrow \mathbb{R}$ be a well defined function.
Let $x = a$ (or simply a) be a real number such that there is some positive real number β so that $(a - \beta, a + \beta) \subseteq D$.
Let $y = f(a)$ we say the function f is continuous at $(a, f(a))$ if and only if
$\forall \epsilon > 0 \quad \exists \delta > 0 \ni |f(x) - f(a)| < \epsilon$ whenever it is true that $|x - a| < \delta$.

Definition 2.2: Let $f : D \longrightarrow \mathbb{R}$ be a well defined function.
Let $x = a$ (or simply a) be a real number such that there is some positive real number β so that $(a - \beta, a + \beta) \subseteq D$.
We say the function f is NOT continuous at $(a, f(a))$ if and only if
$\exists \epsilon > 0$ such that $\forall \delta > 0$ it is the case that $|x - a| < \delta \not\Rightarrow |f(x) - f(a)| < \epsilon$ which means there is some x_1 where $x_1 \in (a - \delta, a + \delta)$ so that forces $|x_1 - a| < \delta$ but it is the case that $|f(x_1) - f(a)| \geq \epsilon$

§2 The Naïve Concept of Continuity of a Function at the Point$(a, f(a))$

Conceptual Definition 2.3: Let $f : D \longrightarrow \mathbb{R}$ be a well defined function.
Let $x = a$ (or simply a) be a real number such that there is some positive real number b so that $(a - b, a + b) \subseteq D$.
Let $y = f(a)$ we say the function f is continuous at $x = a$ if and only if

(1) $f(a)$ exists (is a real number or better yet the point $(a, f(a))$ is a point of the function f);
(2) $\lim_{x \to a^-} f(x) = f(a)$; and,
(3) $\lim_{x \to a^+} f(x) = f(a)$

Examples of the Naïve Concept of Continuity of a Function at $x = a$

The idea is that:
the function exists at $x = a$ meaning $(a, f(a))$ is a point of the function
let $b \in \mathbb{R}$ be positive and let there is the sequence, $\{x_n\}_{n=1}^{\infty} \subseteq (a - b, a)$ is
'any old sequence "crunching" toward a from the left'and $\{x_n\}_{n=1}^{\infty} \longrightarrow a$
whilst there is the sequence, $\{x_m\}_{m=1}^{\infty} \subseteq (a, a + b)$ is
'any old sequence "crunching" toward a from the right'and $a \longleftarrow \{x_m\}_{n=1}^{\infty}$
and both $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow f(a)$ and
$\{f(x_m)\}_{n=1}^{\infty} \longrightarrow f(a)$

So, we mean that $\lim\limits_{x \to a^-} f(x) = f(a) = \lim\limits_{x \to a^+} f(x)$

Example 2.1: Let $f : D \longrightarrow \mathbb{R}$ be such that $D = (0, \infty)$ and $f(x) = \sqrt[3]{x + 2}$.
We claim $f(x) = \sqrt[3]{x + 2}$ is continuous at $x = 25$.

First we note that $f(25) = \sqrt[3]{25 + 2} = \sqrt[3]{27} = 3$

Next we 'crunch' numbers, first, toward 25 from the left and notice the values:

24	$f(24) = \sqrt[3]{26} \approx 2.9624960684$
24.9	$f(24.9) = \sqrt[3]{26.9} \approx 2.99629171439$
24.99	$f(24.99) = \sqrt[3]{26.99} \approx 2.9996295838954$
24.999	$f(24.999) = \sqrt[3]{26.999} \approx 2.9999629625057$
$\downarrow$	$\downarrow$
25^-	3

then from the right:

26	$f(24) = \sqrt[3]{28} \approx 3.0365889718$
25.1	$f(25.1) = \sqrt[3]{27.1} \approx 3.003699$
25.01	$f(25.01) = \sqrt[3]{27.01} \approx 3.00037$
25.001	$f(25.001) = \sqrt[3]{27.001} \approx 3.000037037$
$\downarrow$	$\downarrow$
25^+	3

So, we say yes the function is continuous at $x = 25$

$$\lim_{x \to 25} \sqrt[3]{x + 2} = 3 = f(25)$$

Example 2.2: Let $g : D \longrightarrow \mathbb{R}$ be such that $D = (-\infty, 1) \cup (1, 2) \cup (2, \infty)$ and

$$g(x) = \frac{x^2 - 1}{x^2 - 3x + 2}$$

Notice $1 \notin D$ so, it can not be the case that g is continuous at $x = 1$.
So what about the fact that $\lim\limits_{x \to 1} g(x) = -2$?

It doesn't matter - - without existence of a point at $x = a$ continuity at $x = a$ cannot be!

Examples of the Naïve Concept of Continuity of a Function at $x = a$

Example 2.3: Consider $k : \mathbb{R} \longrightarrow \mathbb{R}$ where

$$k(x) = \begin{cases} 3x + 15, & x < -2 \\ 3, & x = -2 \\ 9, & -2 < x < 1 \\ \frac{4}{(x+2)^2}, & 1 \leqslant x < 3 \\ (x+1)^3, & x \geqslant 3 \end{cases}$$

Notice $\lim\limits_{x \to -2^-} k(x)$ we get 9
Notice $\lim\limits_{x \to -2^+} k(x)$ we get 9
Notice $k(-2) = 3$
Since $9 \neq 3$
it is not the case that k is continuous at $x = -2$

Notice $\lim\limits_{x \to 1^-} k(x)$ we get 9
Notice $\lim\limits_{x \to 1^+} k(x)$ we get $\frac{4}{9}$
Notice $k(1) = \frac{4}{9}$
Since $9 \neq \frac{4}{9}$
it is not the case that k is continuous at $x = 1$

Notice $\lim\limits_{x \to 3^-} k(x)$ we get $\frac{4}{25}$
Notice $\lim\limits_{x \to 3^+} k(x)$ we get 64
Notice $k(3) = 64$ Since $\frac{4}{25} \neq 64$ it is not the case that k is continuous at $x = 3$

Example 2.4: Consider $j : \mathbb{R} \longrightarrow \mathbb{R}$ where

$$k(x) = \begin{cases} 3x + 15, & x < -2 \\ 3, & x = -2 \\ 9, & -2 < x < 1 \\ \frac{81}{(x+2)^2}, & 1 \leqslant x < 3 \\ \frac{2}{3}x + \frac{31}{25}, & x \geqslant 3 \end{cases}$$

Notice $\lim\limits_{x \to -2^-} j(x)$ we get 9
Notice $\lim\limits_{x \to -2^+} j(x)$ we get 9
Notice $j(-2) = 3$
Since $9 \neq 3$
it is not the case that j is continuous at $x = -2$

Notice $\lim\limits_{x \to 1^-} j(x)$ we get 9
Notice $\lim\limits_{x \to 1^+} j(x)$ we get 9
Notice $k(1) = 9$

Since all three values are equal it is the case that j is continuous at $x = 1$

Notice $\lim_{x \to 3^-} j(x)$ we get $\frac{81}{25}$
Notice $\lim_{x \to 3^+} j(x)$ we get $\frac{81}{25}$
Notice $j(3) = \frac{81}{25}$ and since all three values are equal it is the case that j is continuous at $x = 3$

Tasks for you, the student:

(1) Draw the function f in example 2.1 and connect the geometry (graph) to the algebra (the bulldozer);
(2) Draw the function g in example 2.2 and connect the geometry (graph) to the algebra;
(3) Draw the function k in example 2.1 and connect the geometry (graph) to the algebra; and,
(4) Draw the function j in example 2.1 and connect the geometry (graph) to the algebra.

§3 Some *Very* Interesting Examples of the Naïve Concept of Continuity of a Function at $x = 0$

Example 2.5: Consider $D_1 : \mathbb{R} \longrightarrow \mathbb{R}$ where

$$D_1(x) = \begin{cases} 1, & x < 0 \\ 0, & x \geqslant 0 \end{cases}$$

Is it is the case that D_1 is continuous at $x = 0$?

Example 2.6: Consider $D_2 : \mathbb{R} \longrightarrow \mathbb{R}$ where

$$D_2(x) = \begin{cases} x, & x \leq 0 \\ 0, & x > 0 \end{cases}$$

Is it is the case that D_2 is continuous at $x = 0$?

Example 2.7: Consider $D_3 : \mathbb{R} \longrightarrow \mathbb{R}$ where

$$D_3(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{I} \end{cases}$$

Is it is the case that D_3 is continuous at $x = 0$?

Example 2.8: Consider $D_4 : \mathbb{R} \longrightarrow \mathbb{R}$ where

$$D_4(x) = \begin{cases} \frac{1}{m}, & x \in \mathbb{Q} \land x \neq 0 \land m > 0 \land x = \frac{p}{m} \land gcd(p, m) = 1 \\ 0, & x \in \mathbb{I} \lor x = 0 \end{cases}$$

Is it is the case that D_4 is continuous at $x = 0$?

§4 Special Cases Continuity of a Function at $x = a$

Conceptual Definition 4.1: Let $f : D \longrightarrow \mathbb{R}$ be a well defined function such that $D = [b, c]$ where $b, c \in \mathbb{R} \ni b < c$.
Let $x = b$ (or simply b) be a real number such that there is some positive real number a so that $[b, a+b) \subseteq D$.
Let $y = f(b)$ we say the function f is right continuous at $x = b$ if and only if

(1) $f(b)$ exists (is a real number or better yet the point $(b, f(b))$ is a point of the function f) and
(2) $\lim_{x \to b^+} f(x) = f(b)$

Conceptual Definition 4.2: Let $f : D \longrightarrow \mathbb{R}$ be a well defined function such that $D = [b, c]$ where $b, c \in \mathbb{R} \ni b < c$.
Let $x = c$ (or simply c) be a real number such that there is some positive real number a so that $(c-a, c] \subseteq D$.
Let $y = f(c)$ we say the function f is left continuous at $x = c$ if and only if

(1) $f(b)$ exists (is a real number or better yet the point $(b, f(b))$ is a point of the function f) and
(2) $\lim_{x \to c^-} f(x) = f(c)$

§5 Smooth Functions

Recall:
Conceptual Definition' 4.1: Consider our universe $\mathbb{R} \times \mathbb{R}$ which is the Cartesian plane.
Let $\mathcal{C}$ be a curve in the plane. We say **the curve $\mathcal{C}$ is planarly connected (or just connected for short)** if and only if for each two points, p and q, of $\mathcal{C}$ there is an arc 'connecting 'them that is a subset of $\mathcal{C}$. In a vernacular sense, we say planarly connected curve $\mathcal{C}$ has no jumps, no holes, no vertical asymptotes, etc. which 'cut up' the curve.
Now let us build on that:
Conceptual Definition 4.3: Consider our universe $\mathbb{R} \times \mathbb{R}$ which is the Cartesian plane.
Let $\mathcal{C}$ be a curve in the plane. We say **the curve $\mathcal{C}$ is smooth** if and only if for each two points, p and q, of $\mathcal{C}$ there is an arc 'connecting 'them that is a subset of $\mathcal{C}$ and the arc contains no 'sharp points' (immediate directional change). In a vernacular sense, we say the smooth curve $\mathcal{C}$ has no jumps, no holes, no vertical asymptotes, and no sharp points, etc. which 'cut up' the curve or make the curve 'sharply turn.'
Example 4.3-1: Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the well defined function $f(x) = x^2$. It is the case that f is a smooth curve in the plane.

Example 4.3-2: Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the well defined function $f(x) = |x|$. It is the case that f is **not** a smooth curve in the plane.

Conceptual Definition 4.4: Consider our universe $\mathbb{R} \times \mathbb{R}$ which is the Cartesian plane.
Let $\mathcal{C}$ be a curve in the plane. We say **the curve $\mathcal{C}$ is smooth from b to c** (where $b < c \wedge [b, c] \subseteq \mathcal{C}$ if and only if for each two points, p and q, between b and c there is an arc 'connecting 'them that is a subset of $[b, c]$ and the arc contains no 'sharp points' (immediate directional change). In a vernacular sense, we say smooth part of the curve $\mathcal{C}$ from b to c has no jumps, no holes, no vertical asymptotes, and no sharp points, etc. which 'cut up' the curve or make the curve 'sharply turn.'