Handout I
The Concept of Limit of a Function at $x = a$
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Let our universe be $\mathbb{R} \times \mathbb{R}$ which is the Cartesian plane.   Let $D \subseteq \mathbb{R}$

Definition 1.1A: Let $f : D \longrightarrow \mathbb{R}$ be a well defined function.
Let $x = a$ (or simply a) be a real number such that there are some positive real numbers b and c so that $(a - b, a) \subseteq D$ and $(a + c, a) \subseteq D$.
Let $y = l$ (or we just say the real number l).
We say the limit of $f(x)$ at $x = a$ is $y = l$ if and only if
for each lines $y = m$ and $y = p$ where $m < l < p$
there exist lines $x = h$ and $x = k$ where $h < a < k$
so that for each $x \in D$ that is also in (h, a) or (a, k)
it is the case that $f(x) \in (m, p)$.

The notation for this is $\lim_{x\to a} f(x) = l$, $lim_{x\to a}(f(x)) = l$, or $f(x) \stackrel{x\to a}{\longrightarrow} l$.

Definition 1.1B: Let $f : D \longrightarrow \mathbb{R}$ be a well defined function.
Let $x = a$ (or simply a) be a real number such that there is some positive real number β so that $(a - \beta, a) \cup (a, a + \beta) \subseteq D$.
Let $y = L$ (or we just say the real number L).
We say the limit of $f(x)$ at $x = a$ is $y = L$ if and only if
$\forall \epsilon > 0 \quad \exists \delta > 0 \ni$
$0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \epsilon$.

Definition 1.1C: Let $f : D \longrightarrow \mathbb{R}$ be a well defined function.
Let $x = a$ (or simply a) be a real number such that there is some positive real number β so that $(a - \beta, a) \cup (a, a + \beta) \subseteq D$.
Let $y = L$ (or we just say the real number L).
We say the limit of $f(x)$ at $x = a$ is $y = L$ if and only if
$\forall \epsilon > 0 \quad \exists \delta > 0 \ni$
$|f(x) - L| < \epsilon$ whenever it is true that $0 < |x - a| < \delta$.

Definition 1.2: Let $f : D \longrightarrow \mathbb{R}$ be a well defined function.
Let $x = a$ (or simply a) be a real number such that there is some positive real number β so that $(a - \beta, a) \cup (a, a + \beta) \subseteq D$.
We say the limit of $f(x)$ at $x = a$ *does not exist* (**DNE** for short) if and only if
$\exists \epsilon > 0$ such that $\forall \delta > 0$ it is the case that $0 < |x - a| < \delta \not\Rightarrow |f(x) - L| < \epsilon$ which means there is some $x_1 \neq a$ where $x_1 \in (a - \delta, a + \delta)$ so that forces $0 < |x_1 - a| < \delta$ but it is the case that $|f(x_1) - L| \geq \epsilon$

For all of the above the notation for limit when it exists and is l is:

$$\lim_{x\to a} f(x) = l$$

and when it does not exist the notation and shorthand should be written as:

$$\lim_{x\to a} f(x) \quad DNE$$

The Naïve Concept of Limit of a Function at $x = a$

Conceptual Definition 1.3: Let $f : D \longrightarrow \mathbb{R}$ be a well defined function. Let $x = a$ (or simply a) be a real number such that there are some positive real numbers b and c so that $(a - b, a) \subseteq D$ and $(a + c, a) \subseteq D$.
Let $y = l$.
Let the sequence, $\{x_n\}_{n=1}^{\infty} \subseteq (a - b, a)$ be 'any old sequence "crunching" toward a from the left'
and $\{x_n\}_{n=1}^{\infty} \longrightarrow a$
We say the 'left limit' of $f(x)$ for $x \longrightarrow a^-$ is $y = l$ if and only if $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow l$

The notation for this is $\lim_{x \to a^-} f(x) = l$

Conceptual Definition 1.4: Let $f : D \longrightarrow \mathbb{R}$ be a well defined function. Let $x = a$ (or simply a) be a real number such that there are some positive real numbers b and c so that $(a - b, a) \subseteq D$ and $(a + c, a) \subseteq D$.
Let $y = l$.
Let the sequence, $\{x_n\}_{n=1}^{\infty} \subseteq (a, a + c)$ be 'any old sequence "crunching" toward a from the right'
and $\{x_n\}_{n=1}^{\infty} \longrightarrow a$
We say the 'right limit' of $f(x)$ for $x \longrightarrow a^+$ is $y = l$ if and only if $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow l$

The notation for this is $\lim_{x \to a^+} f(x) = l$

Conceptual Definition 1.5: Let $f : D \longrightarrow \mathbb{R}$ be a well defined function. Let $x = a$ (or simply a) be a real number such that there are some positive real numbers b and c so that $(a - b, a) \subseteq D$ and $(a + c, a) \subseteq D$.
Let $y = l$.
We say the limit of $f(x)$ is l if and only if
$\lim_{x \to a^-} f(x) = l$ and
$\lim_{x \to a^+} f(x) = l$

Examples of Naïve Concept of Limit of a Function at $x = a$

The idea is that there is the sequence, $\{x_n\}_{n=1}^{\infty} \subseteq (a - b, a)$ is
'any old sequence "crunching" toward a from the left'and $\{x_n\}_{n=1}^{\infty} \longrightarrow a$
whilst there is the sequence, $\{x_m\}_{m=1}^{\infty} \subseteq (a, a + c)$ is
'any old sequence "crunching" toward a from the right'and $a \longleftarrow \{x_m\}_{n=1}^{\infty}$
and both $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow l$ and
$\{f(x_m)\}_{n=1}^{\infty} \longrightarrow l$

Example 1.1: Let $f : D \longrightarrow \mathbb{R}$ be such that $D = (0, \infty)$ and $f(x) = \sqrt[3]{x + 2}$.
We claim $\lim_{x \to 25} f(x) = 3$.
We show this by 'crunching numbers, first, toward 25 from the left and notice the values:
24 $\quad f(24) = \sqrt[3]{26} \approx 2.9624960684$
24.9 $\quad f(24.9) = \sqrt[3]{26.9} \approx 2.99629171439$
24.99 $\quad f(24.99) = \sqrt[3]{26.99} \approx 2.9996295838954$
24.999 $\quad f(24.999) = \sqrt[3]{26.999} \approx 2.9999629625057$
$\downarrow \qquad \downarrow$
$25^- \qquad 3$

then from the right:
26 $\quad f(24) = \sqrt[3]{28} \approx 3.0365889718$
25.1 $\quad f(25.1) = \sqrt[3]{27.1} \approx 3.003699$
25.01 $\quad f(25.01) = \sqrt[3]{27.01} \approx 3.00037$
25.001 $\quad f(25.001) = \sqrt[3]{27.001} \approx 3.000037037$
$\downarrow \qquad \downarrow$
$25^+ \qquad 3$

So, we say

$$\lim_{x\to 25} \sqrt[3]{x+2} = 3$$

Example 1.2: Let $f : D \longrightarrow \mathbb{R}$ be such that $D = (-\infty, 1) \cup (1, 2) \cup (2, \infty)$ and $f(x) = \frac{x^2-1}{x^2-3x+2}$.
We claim $\lim_{x\to 1} f(x) = -2$.
Notice $1 \notin D$ but we can still try to find a limit for $f(x)$ at $x = 1$.
We show this by 'crunching numbers, first, toward 1 from the left and notice the values:
$0 \qquad f(0) = -\frac{1}{2}$
$\frac{9}{10} \qquad f(\frac{9}{10}) = -\frac{19}{11}$
$\frac{99}{100} \qquad f(\frac{99}{100}) = -\frac{199}{101}$
$\downarrow \qquad \downarrow$
$1^- \qquad -2$

$0 \qquad f(2)$ does not exist
$\frac{11}{10} \qquad f(\frac{11}{10}) = -\frac{21}{11}$
$\frac{101}{100} \qquad f(\frac{101}{100}) = -\frac{201}{101}$
$\downarrow \qquad \downarrow$
$1^+ \qquad -2$

So, we say

$$\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2} = -2$$

Example 1.3: Consider $k : \mathbb{R} \longrightarrow \mathbb{R}$ where

$$k(x) = \begin{cases} 3x+15, & x < -2 \\ 3, & x = -2 \\ 9, & -2 < x < 1 \\ \frac{4}{(x+2)^2}, & 1 \leqslant x < 3 \\ (x+1)^3, & x \geqslant 3 \end{cases}$$

When we consider $\lim_{x\to 3^-} k(x)$ we get $\frac{4}{25}$

When we consider $\lim_{x\to 3^+} k(x)$ we get 64
Since $\frac{4}{25} \neq 64$ $\lim_{x\to 3} k(x)$ **DNE**.
When we consider $\lim_{x\to -2^-} k(x)$ we get 9

When we consider $\lim_{x\to -2^+} k(x)$ we get 9
Since $9 = 9$ $\lim_{x\to 3} k(x) = 9$. Please notice $f(-2)$ is not 9. Wonderful! Notice that limits at $x = a$ do not

depend on function value at $x = a$.

Example 1.4: Let $D = [0, \infty)$. Consider $j : D \longrightarrow \mathbb{R}$ where

$$j(x) = \begin{cases} 3x - 44, & 0 \leqslant x < 4 \\ 4 - 9x, & 4 < x < 10 \\ \sin x, & x \geqslant 10 \end{cases}$$

When we try to consider $\lim_{x \to 0^-} j(x)$ we get nothing - it doesn't exist since there are no values of D in $(-\infty, 0)$. End of story.
$\lim_{x \to 3} j(x)$ **DNE**.

When we consider $\lim_{x \to 4^-} j(x)$ we get -32

When we consider $\lim_{x \to 4^+} j(x)$ we get -32
Since $-32 = -32$
$\lim_{x \to 4} j(x) = -32$!

When we consider $\lim_{x \to 10^-} j(x)$ we get -86

When we consider $\lim_{x \to 10^+} j(x)$ we get $\sin 10$
Since $-86 \neq \sin 10$ the limit does not exist for $j(x)$ at $x = 10$.