Handout III

The Concept of Derivatives of a Function at the Point $(a, f(a))$; the Concept of Derivatives of a Function as A Function; and, the Graphical Concept of Derivatives of a Function

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SPRING 2009

Let our universe be $\mathbb{R} \times \mathbb{R}$ which is the Cartesian plane. Let $D \subseteq \mathbb{R}$

You will notice that derivative of $f(x)$ at $(a, f(a))$ is just a 'tad' more developed than limit or continuity and is built upon those concepts.

Definition 3.1: Let $f : D \longrightarrow \mathbb{R}$ be a well defined function.
Let $x = a$ (or simply a) be a real number such that there is some positive real number b so that $(a-b, a+b) \subseteq D$.
We say the function f has a derivative at a if and only if
it is the case that $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists.

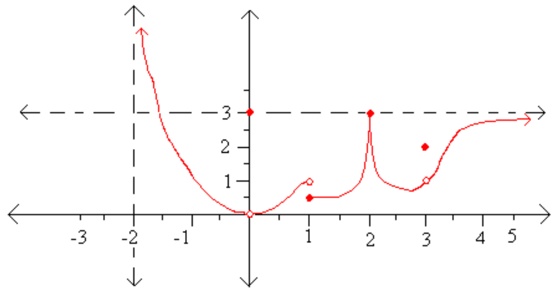
Definition 3.2: Let $f : D \longrightarrow \mathbb{R}$ be a well defined function.
We say the function f has a derivative which is a function of x also, call it $f'(x)$ if and only if
it is the case that $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ exists.

Consequence of Definition 3.2: $f'(x) : A \longrightarrow \mathbb{R}$ is a well defined function from the set A to the set $\mathbb{R}$ where $A \subseteq D$.

Example 3.1: Let $f : D \longrightarrow \mathbb{R}$ be such that $D = (0, \infty)$ and $f(x) = \sqrt{x+2}$.
We claim $f'(x)$ exists $\forall x \in D$.
Note $f(x) = \sqrt{x+2}$.
So, $f(x+h) = \sqrt{(x+h)+2}$.
Thus, $\lim\limits_{h \to 0} \dfrac{\sqrt{(x+h)+2} - \sqrt{x+2}}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt{x+h+2} - \sqrt{x+2}}{h} =$
$\lim\limits_{h \to 0} \left(\dfrac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \right) \cdot \left(\dfrac{\sqrt{x+h+2} + \sqrt{x+2}}{\sqrt{x+h+2} + \sqrt{x+2}} \right) =$
$\lim\limits_{h \to 0} \left(\dfrac{(x+h+2) - (x+2)}{h \cdot (\sqrt{x+h+2} + \sqrt{x+2})} \right) =$
$\lim\limits_{h \to 0} \left(\dfrac{h}{h \cdot (\sqrt{x+h+2} + \sqrt{x+2})} \right) =$
$\lim\limits_{h \to 0} \left(\dfrac{1}{(\sqrt{x+h+2} + \sqrt{x+2})} \right) =$
$\left(\dfrac{1}{(\sqrt{x+0+2} + \sqrt{x+2})} \right) =$
$\left(\dfrac{1}{2 \cdot \sqrt{x+2}} \right) = \dfrac{1}{2}(x+2)^{-1/2}$

$$f(x) = \sqrt{x+2} \Longrightarrow f'(x) = \frac{1}{2}(x+2)^{-1/2}$$

<u>Example 3.2:</u> Let $g : D \longrightarrow \mathbb{R}$ where $D = (-2, \infty)$ be defined by the following graph. If we consider the graph of the following



we note that $g(x)$ is well defined $\forall x \in D$.
We note that $g(x)$ has a vertical asymptote at $x = -2$. It should be noted that $\lim\limits_{x \to -2^-} g(x)$ does not exist in the "$+\infty$" sense.
We note that $g(x)$ has a horizontal asymptote at $y = 3$. $\lim\limits_{x \to \infty} g(x) = 3$ but there is NOT a $\lim\limits_{x \to -\infty} g(x)$ since there is not even a function $\forall x \in (-\infty, -2]$.
Now as to limits, continuity, and derivatives, note:
For $x = 0$:
$\lim\limits_{x \to 0} g(x) = 0$; $g(x)$ is not continuous at $x = 0$ since $g(0) = 3$ but $\lim\limits_{x \to 0} g(x) = 0$; and, $g\prime(x)$ does not exist at $x = 0$.
For $x = 1$:
$\lim\limits_{x \to 1} g(x)$ does not exist since $\lim\limits_{x \to 1^-} g(x) = 1 \quad \wedge \quad \lim\limits_{x \to 1^+} g(x) = \dfrac{1}{2}$; $g(x)$ is not continuous at $x = 1$ since the limit does not exist; and, $g\prime(x)$ does not exist at $x = 1$.
For $x = 2$:
$\lim\limits_{x \to 2} g(x) = 3$; $g(x)$ is continuous at $x = 2$ since the limit is equal to $g(2) = 3$; but, $g\prime(x)$ does not exist at $x = 2$ (there is a vertical tangent at $x = 2$ which means the slope of the tangent line.
For $x = 3$:
$\lim\limits_{x \to 0} g(x) = 1$; $g(x)$ is not continuous at $x = 0$ since $g(3) = 2$ but $\lim\limits_{x \to 0} g(x) = 1$; and, $g\prime(x)$ does not exist at $x = 3$.
is not a real number (does not exist).
For $p \in D \ni p \neq 0, p \neq 1, p \neq 2, p \neq 3$:
$\lim\limits_{x \to p} g(x)$ exists; $g(x)$ is continuous at $x = p$ since $g(p) = \lim\limits_{x \to p} g(x)$; and, $g\prime(x)$ exists at $x = p$.

So, we should clearly realise that:
<u>Theorem 3.1:</u> Let $f : D \longrightarrow \mathbb{R}$ be a well defined function.
Let f have a derivative at a,
therefore it must be the case that $f(x)$ is continuous at $x = a$; and, furthermore,
$\lim\limits_{x \to a} f(x)$ exists.

Meaning:
the existence of a derivative at a point $\Longrightarrow$ continuous at the point $\Longrightarrow$ the limit exists at the point.