MATH 140
DR. MCLOUGHLIN'S CLASS BERNOULLI AND BINOMIAL RANDOM VARIABLES
HANDOUT III

Note: The basic building block of probability is set theory:
Suppose we have a well defined sample space S (a well defined universe *U* in set theory 'lingo') and events E_1, E_2, etc. (sets E_1, E_2, etc.), yada, yada, yada. The basic ideas are grounded in the sets!

Definition 1: If X is a discrete random variable, the function given by $f(x) = \Pr(X = x)$ for each x in the domain of the function is called the probability mass function (p. m. f.).

Theorem 1: A function serves as a p. m. f. of a discrete random variable iff its values $f(x)$ satisfy both:

1. $f(x) \geq 0 \;\forall\; x \in \text{dom}(f)$ and 2. $\sum_x f(x) = 1$.

Definition 2: If X is a discrete random variable and the function given by $f(x) = \Pr(X = x)$ for each x in the domain of the function is the p. m. f. at x, then the expected value (or mean) of X is $E[X] = \sum_x x \cdot f(x)$. $E[X] = \mu$

Definition 3: If X is a discrete random variable and the function given by $f(x) = \Pr(X = x)$ for each x in the domain of the function is the p. m. f. at x, then the r^th^ moment about the origin of X is $E[X^r] = \sum_x x^r \cdot f(x)$. $E[X^r] = \mu_r'$

Definition 4: If X is a discrete random variable and the function given by $f(x) = \Pr(X = x)$ for each x in the domain of the function is the p. m. f. at x, then the variance (or second moment about the mean) of X is $\text{Var}[X] = \sum_x (x - E[X])^2 \cdot f(x)$.

$\text{Var}[X] = \sigma^2$ $\text{Var}[X] = E[(X - \mu)^2]$

Definition 5: If X is a discrete random variable and the function given by $f(x) = \Pr(X = x)$ for each x in the domain of the function is the p. m. f. at x, then the standard deviation of X is

$SD[X] = \sqrt{\sum_x (x - E[X])^2 \cdot f(x)}$. $SD[X] = \sigma$

Definition 6: If X is a discrete random variable and the function given by $f(x) = \Pr(X = x)$ for each x in the domain of the function is the p. m. f. at x, then the r^{th} moment about the mean of X is $E[(X - \mu)^r] = \sum_x (x - E[X])^r \cdot f(x)$. $E[(X - \mu)^r] = \mu_r$

<u>Theorem 2</u>: If X is a discrete random variable and the function given by *f*(x) = Pr(X = x) for each x in the domain of the function is the p. m. f. at x, then Var[X] = $\mu_2' - \mu^2$ = $E[X^2] - (E[X])^2$

The Bernoulli trial is a probabilistic (or stochastic) experiment that can have one of two outcomes, success (X = 1) or failure (X = 0) in which the probability of success is p. The parametre is p.

$p \in (0, 1)$
$x \in \{0, 1\}$

$$\text{Ber}(x, p) = \Pr(X = x) = \begin{cases} p, & x = 1 \\ 1 - p & x = 0 \\ 0 & \text{else} \end{cases}$$

$E[X] = \mu = p$

$\mu_r' = p \quad \forall\, r \in \mathbb{N}$

$\text{Var}[X] = \sigma^2 = p(1 - p)$
$\text{SD}[X] = \sigma = \sqrt{p(1-p)}$

The Binomial variate is the number of successes in n-independent Bernoulli trials where the probability of success at each trial is p. The parametres are p and n(the number of trials).

$p \in (0, 1)$

$n \in \mathbb{N}$

$x \in \{0, 1, 2, \ldots, (n - 1), n\}$

$$\text{Bin}(x, p, n) = \Pr(X = x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0,1,2,\ldots,n \\ 0 & \text{else} \end{cases}$$

$\mu = np$

$\mu_2' = np(np + (1 - p))$

$\mu_3' = np((n-1)(n-2)p^2 + 3p(n - 1) + 1)$

$\sigma^2 = np(1 - p)$

$\mu_3 = np(1 - p)((1 - p) - p)$

$\mu_4 = np((1 + 3p(1-p)(n - 2))$

Examples of Binomial random variables:

1. Flip a balanced coin 5 times and record the number of heads. Let X be the number of heads obtained. Then

$X \sim j(x)$ where $j(x)$ a probability mass function such that $j(x) = \begin{cases} \binom{5}{x} \cdot \left(\frac{1}{2}\right)^{x} \cdot \left(\frac{1}{2}\right)^{(5-x)} & x \in \mathbb{N}_5^* \\ 0 & \text{else} \end{cases}$

X = x	0	1	2	3	4	5	else
Pr(X = x)	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$	0

2. Flip an unbalanced coin 5 times and record the number of heads.

So, $X \sim f(x)$ where $f(x)$ a probability mass function such that $f(x) = \begin{cases} \binom{5}{x} \cdot \left(\frac{1}{3}\right)^{x} \cdot \left(\frac{2}{3}\right)^{(5-x)} & x \in \mathbb{N}_5^* \\ 0 & \text{else} \end{cases}$

X = x	0	1	2	3	4	5	else
Pr(X = x)	$\frac{1}{243}$	$\frac{10}{243}$	$\frac{40}{243}$	$\frac{80}{243}$	$\frac{80}{243}$	$\frac{32}{243}$	0

3. A student takes a multiple choice test with 12 question; each question has five options for a response; each question has one and only one option correct; and, a response on a question has no effect on a response on any or all of the other 11 questions (all questions are statistically independent, pair-wise, three-wise, etc, 12-wise)

So, $X \sim g(x)$ where $g(x)$ a probability mass function such that $g(x) = \begin{cases} \binom{12}{x} \cdot \left(\frac{1}{5}\right)^{x} \cdot \left(\frac{4}{5}\right)^{(12-x)} & x \in \mathbb{N}_{12}^* \\ 0 & \text{else} \end{cases}$

X = x	0	1	2	3	4	5	6
Pr(X = x)	$\frac{16777216}{244140625}$	$\frac{50331648}{244140625}$	$\frac{69206016}{244140625}$	$\frac{57671680}{244140625}$	$\frac{32440320}{244140625}$	$\frac{12976128}{244140625}$	$\frac{3784704}{244140625}$

7	8	9	10	11	12	else
$\frac{811008}{244140625}$	$\frac{126720}{244140625}$	$\frac{14080}{244140625}$	$\frac{1056}{244140625}$	$\frac{48}{244140625}$	$\frac{1}{244140625}$	0

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