MATH 100
DR. MCLOUGHLIN'S HANDY DANDY
GRAPHING GUIDE USING POSITIVE NEGATIVE ANALYSIS PART I

Recall, we discussed basic graphing techniques of functions in Part I and Part II of the Handy Dandy Systematic Graphing Guide. It only works for functions that are in proper polynomial form like $g(x) = -6(x-1)^7 + 1$ where g: $\mathbb{R} \longrightarrow \mathbb{R}$ and doesn't work for functions of the type like $f(x) = \dfrac{(x-1)^2 \cdot (x+2)}{(x-3)}$ where f: $(-\infty, 3) \cup (3, \infty) \longrightarrow \mathbb{R}$ let alone functions of the type like $h(x) = \dfrac{\sqrt{2x-1}}{x+3}$ where h: $[\frac{1}{2}, \infty) \longrightarrow \mathbb{R}$ or even the polynomial $k(x) = x^5 + 4x^4 - 25x^3 - 16x^2 + 84x$ where k: $\mathbb{R} \longrightarrow \mathbb{R}$

Example 1: Graph the polynomial $k(x) = x^5 + 4x^4 - 25x^3 - 16x^2 + 84x$ where k: $\mathbb{R} \longrightarrow \mathbb{R}$ using positive-negative analysis (PNA).

So, what to do?

Well, we have much we can accomplish.

We an see it is a squig that to the extreme right goes up and at the extreme left goes up because the basic graph is quintic so it is a squig.

We can try to factor.
$k(x) = x^5 + 4x^4 - 25x^3 - 16x^2 + 84x$ where k: $\mathbb{R} \longrightarrow \mathbb{R}$

$x^5 + 4x^4 - 25x^3 - 16x^2 + 84x = x(x^4 + 4x^3 - 25x^2 - 16x + 84)$
that at least tells us a root of the polynomial is zero since x = x – 0
but, now we are stuck. Eh? I can't tell (off hand) how $x^4 + 4x^3 - 25x^2 - 16x + 84$ factors.

Well, we can use Descartes Law of Sign Changes (DLSC) to figure out how many possible real positive or negative roots could exist for polynomials.

For $x^4 + 4x^3 - 25x^2 - 16x + 84$, there are 2 changes so the number of *possible* positive real roots is 2 or 0.
For $(-x)^4 + 4(-x)^3 - 25(-x)^2 - 16(-x) + 84 = x^4 - 4x^3 - 25x^2 + 16x + 84$, there are 2 changes so the number of *possible* negative real roots is 2 or 0.

That at least helps us see there are possibilities.

Now, we can use the Rational Root Theorem (RRT) to figure out the possible rational roots that could exist for this polynomial.

Factors of 1: 84 and 1, 42 and 2, 28 and 3, 21 and 4, 14 and 6, 12 and 7.
Factors of 1: 1 and -1

So, the possible rational roots are:
-84, -42, -28, -21, -14, -12, -7, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, and 84.

Not so bad when you consider there are infinitely many rational numbers and we have limited the list down to 24.

Let us try one say, -10, and see if it is a root of $x^4 + 4x^3 - 25x^2 - 16x + 84$:

-10 |	1	4	-25	-16	84
		-10	60	-350	3660
	1	-6	35	-366	| 3744.

Nope.

Let us try another say, -4, and see if it is a root of $x^4 + 4x^3 - 25x^2 - 16x + 84$:

-4 |	1	4	-25	-16	84
		-4	0	100	-336
	1	0	-25	84	| -252.

Nope, but since for -10 the result was positive and for -4 the result was negative there is a root between -10 and -4!

(we will use this later)

Let us try yet another, -2, and see if it is a root of $x^4 + 4x^3 - 25x^2 - 16x + 84$:

-2 |	1	4	-25	-16	84
		-2	-4	58	-84
	1	2	-29	42	| 0 .

Bingo!

So, $x^4 + 4x^3 - 25x^2 - 16x + 84 = (x+2)\cdot(x^3 + 2x^2 - 29x + 42)$

Hence, back to the original function:

k(x) = $x^5 + 4x^4 - 25x^3 - 16x^2 + 84x$ where *k*: $\mathbb{R} \longrightarrow \mathbb{R}$

$x^5 + 4x^4 - 25x^3 - 16x^2 + 84x$

$= x(x^4 + 4x^3 - 25x^2 - 16x + 84)$

$= x \cdot (x + 2) \cdot (x^3 + 2x^2 - 29x + 42)$

We are making progress.

Now, what about $x^3 + 2x^2 - 29x + 42$?

Notice from the list:
-84, -42, -28, -21, -14, -12, -7, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, and 84.
We can now delete -84, -28, 28, and 84 (why?)

I love 2's so let us try 2 and see if it is a root of $x^3 + 2x^2 - 29x + 42$:

$$\begin{array}{r|rrrr} \underline{2} & 1 & 2 & -29 & 42 \\ & & 2 & 8 & -42 \\ \hline & 1 & 4 & -21 & |\underline{\ 0\ .} \end{array}$$

Yes!!!!!!!!

So, $x^3 + 2x^2 - 29x + 42 = (x - 2) \cdot (x^2 + 4x - 21)$

Hence, back to the original function:

k(x) = $x^5 + 4x^4 - 25x^3 - 16x^2 + 84x$ where *k*: $\mathbb{R} \longrightarrow \mathbb{R}$

$x^5 + 4x^4 - 25x^3 - 16x^2 + 84x = x(x^4 + 4x^3 - 25x^2 - 16x + 84)$

$= x \cdot (x + 2) \cdot (x^3 + 2x^2 - 29x + 42) = x \cdot (x + 2) \cdot (x - 2) \cdot (x^2 + 4x - 21)$

We could use the RRT more, but let us just factor $x^2 + 4x - 21$
It is $x^2 + 4x - 21 = (x + 7)(x - 3)$
So,

$x^5 + 4x^4 - 25x^3 - 16x^2 + 84x = x(x^4 + 4x^3 - 25x^2 - 16x + 84)$

$= x \cdot (x + 2) \cdot (x^3 + 2x^2 - 29x + 42) = x \cdot (x + 2) \cdot (x - 2) \cdot (x^2 + 4x - 21)$

$= x \cdot (x + 2) \cdot (x - 2) \cdot (x + 7) \cdot (x - 3)$

So, *k*(x) = $x^5 + 4x^4 - 25x^3 - 16x^2 + 84x$ where *k*: $\mathbb{R} \longrightarrow \mathbb{R}$ has x-intercepts of

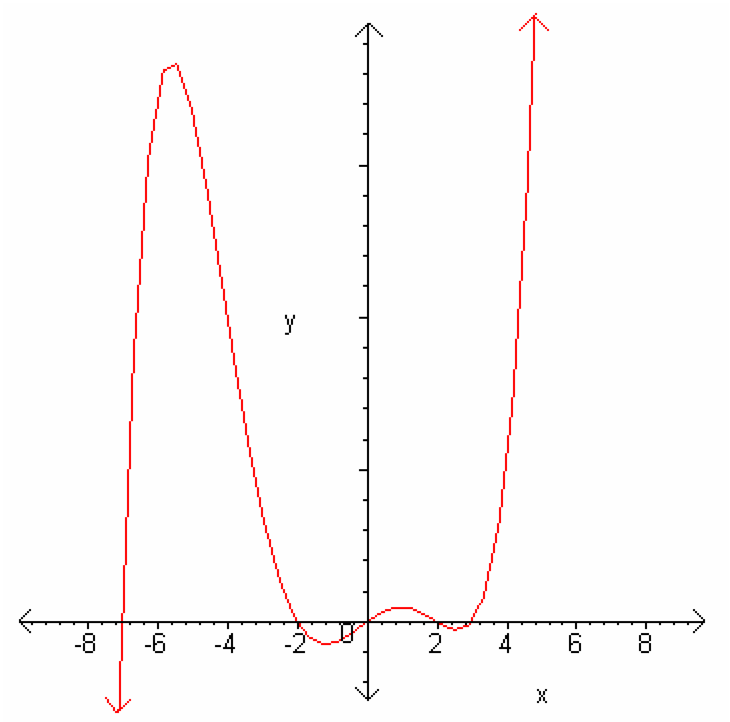
(0, 0), (-2, 0), (2, 0), (-7, 0) (3, 0) because the roots are (in order) -7, -2, 0, 2, 3.

Let's do the PNA with the chart method we discussed earlier in the semestre:

-∞ -7 -2 0 2 3 ∞

	-10	-5	-1	1	$\frac{5}{2}$	5
(x + 7)	–	+	+	+	+	+
(x + 2)	–	–	+	+	+	+
x	–	–	–	+	+	+
(x – 2)	–	–	–	–	+	+
(x – 3)	–	–	–	–	–	+
wtmo[1]	–	+	–	+	–	+
	B[2]	A[3]	B	A	B	A

And so, $k(x) = x^5 + 4x^4 - 25x^3 - 16x^2 + 84x$ where $k\colon \mathbb{R} \longrightarrow \mathbb{R}$ looks like:



What is the main point?

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[1] Wtmo means whole thing multiplied out.

[2] Below the x-axis over (–∞, -7) ∪ (-2, 0) ∪ (2, 3)

[3] Above the x-axis over (-7, -2) ∪ (0, 2) ∪ (3, ∞)

Graphing using PNA involves f*inding where f (x) is above or below the x - axis.*

The process is as follows:

1. Factor both the numerator and denominator of *f* (x).

2. Cancel any common factors, the result will create a hole at the x-value.

3. Set the numerator of *f* (x) = 0 yielding first coordinates of x - intercepts
set the denominator of *f* (x) = 0 yielding vertical asymptotes.

call these values *cut values.*

4. Do a positive - negative Analysis of *f* (x) with the cut values

if $f(x) > 0$, the graph is above the x -axis for all values between the cut values
if $f(x) < 0$, the graph is below the x -axis for all values between the cut values

5. Plug the cut values back into *f* (x). If it exists,
it is the second coordinate of the x - intercept,
if it does not exist, then that cut value should have been the x - value of a vertical asymptote (or if you did not simplify the function, it could be a hole in the graph).

<u>Example 2</u>: Graph the rational function $p(x) = \dfrac{x^2 - 4}{x^2 + 5x - 14}$

where $p: (-\infty, -7) \cup (-7, 2) \cup (2, \infty) \longrightarrow \mathbb{R}$ using positive-negative analysis (PNA).

First notice $\dfrac{x^2 - 4}{x^2 + 5x - 14} = \dfrac{(x-2)\cdot(x+2)}{(x-2)\cdot(x+7)} = \dfrac{(x+2)}{(x+7)}$

so, $p(x)$ will look like $t(x) = \dfrac{(x+2)}{(x+7)}$

where $t: (-\infty, -7) \cup (-7, \infty) \longrightarrow \mathbb{R}$

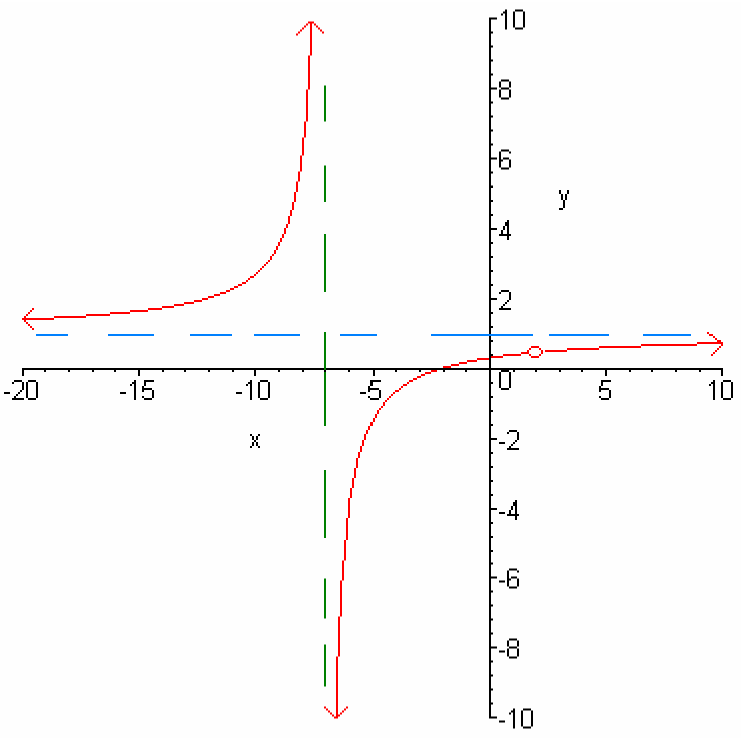
but $t(2) = \dfrac{4}{9}$ hence $p(x)$ will have a hole at $(2, \dfrac{4}{9})$

Notice the cut values are -7 and -2.
Doing the PNA yields:

$-\infty \qquad -7 \qquad -2 \qquad \infty$

	-10	-5	0
(x + 7)	–	+	+
(x + 2)	–	–	+
wtmo[4]	+	–	+
	A	B	A

So, $p(x) = \dfrac{x^2 - 4}{x^2 + 5x - 14}$ where p: $(-\infty, -7) \cup (-7, 2) \cup (2, \infty) \longrightarrow \mathbb{R}$ looks like:



Notice we can tie this back to systematic graphing and look at it another way

Consider p: $(-\infty, -7) \cup (-7, 2) \cup (2, \infty) \longrightarrow \mathbb{R}$.

Notice $\dfrac{x^2 - 4}{x^2 + 5x - 14} = \dfrac{(x-2)\cdot(x+2)}{(x-2)\cdot(x+7)} = \dfrac{(x+2)}{(x+7)}$

so, $p(x)$ will look like $t(x) = \dfrac{(x+2)}{(x+7)}$

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[4] Wtmo means whole thing multiplied out.

where t: $(-\infty, -7) \cup (-7, \infty) \longrightarrow \mathbb{R}$

but $t(2) = \frac{4}{9}$ hence $p(x)$ will have a hole at $(2, \frac{4}{9})$

But consider $(x + 2) \div (x + 7)$:

$$\begin{array}{r|rr} -7 & 1 & 2 \\ & & -7 \\ \hline & 1 & \underline{|\,-5\,.} \end{array}$$

So, $t(x) = \frac{(x+2)}{(x+7)} = 1 - \frac{5}{x+7} = -5(x+7)^{-1} + 1$! Yowsa!

It is a hyperbola (power -1) turned upside down (multiplied by -5) shifted left 7 (the x + 7 inside the parentheses) and shifted up 1 (the +1). All that is left to do is the hole at at $(2, \frac{4}{9})$ since $p(x)$ looks like $t(x)$ with a hole!

Cool, huh?

Example 3: Graph the rational function $f(x) = \frac{(x-1)^2 \cdot (x+2)}{(x-3)}$ where

f: $(-\infty, 3) \cup (3, \infty) \longrightarrow \mathbb{R}$

Notice both the numerator and denominator of $f(x)$ are already factored for us; so, we can also note that there aren't any common factors to create holes.
Now, setting the numerator of $f(x) = 0$ yielding first coordinates of x – intercepts and the denominator of $f(x) = 0$ yielding vertical asymptotes.
So, the *cut values* are 1, -2, and 3. Notice 1 is a root of multiplicity 2, -2 is a root of multiplicity 1 (so the points (1, 0) and (-2, 0) are x-intercepts. Further, notice 3 is a root of the denominator so x = 3 is a vertical asymptote.[5]

4. Do a positive - negative Analysis of $f(x)$ with the cut values

if $f(x) > 0$, the graph is above the x -axis for all values between the cut values
if $f(x) < 0$, the graph is below the x -axis for all values between the cut values

5. Plug the cut values back into $f(x)$. If it exists,

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[5] Contrast this with example 2 where there was a hole a x = 1 but a vertical asymptote at x = -7. If cancellation occurs we get a hole, if no cancellation occurs and there is a polynomial in the denominator, that will yield vertical asymptotes.

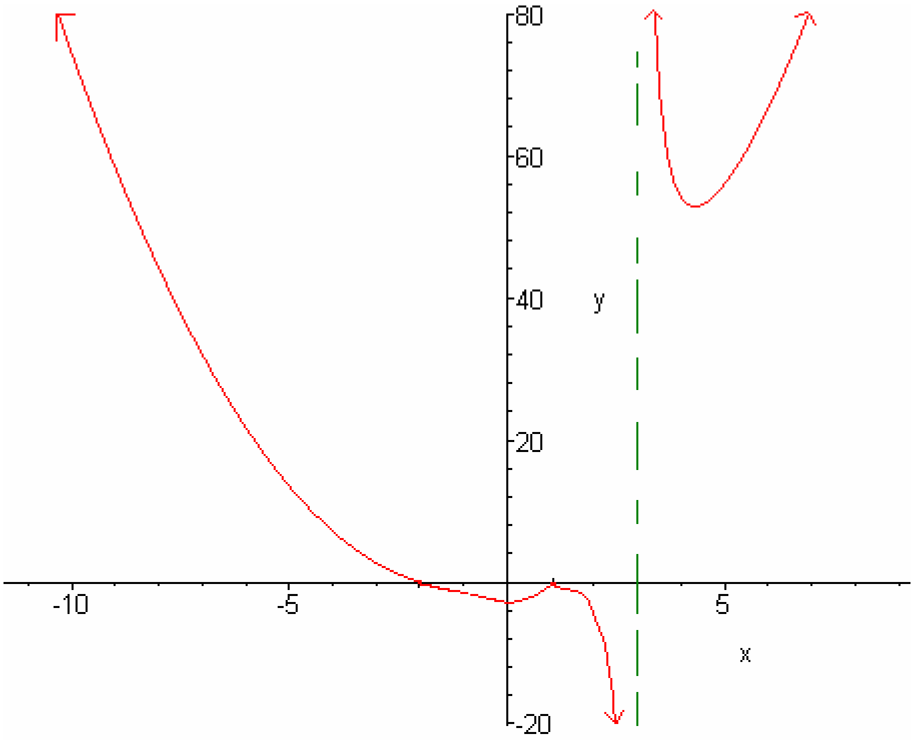
it is the second coordinate of the x - intercept,
if it does not exist, then that cut value should have been the x - value of a vertical asymptote (or if you did not simplify the function, it could be a hole in the graph).

Doing the PNA yields:

-∞ -2 1 3 ∞

	-5	0	2	4
$(x + 2)$	–	+	+	+
$(x - 1)^2$	+	+	+	+
$(x - 3)$	–	–	–	+
wtmo[6]	+	–	–	+
	A	B	B	A

And, the graph:



Exercises:

1. Graph completely using positive-negative analysis the rational function

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[6] Wtmo means whole thing multiplied out.

$f_1(x) = 2x^4 + 5x^3 - 5x^2 - 20x - 12$ where $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$

label all x-intercepts, holes, and vertical asymptotes.

2. Graph completely using positive-negative analysis the rational function

$$f_2(x) = \frac{2x^4 + 5x^3 - 5x^2 - 20x - 12}{x - 2}$$

where $f_2 : (-\infty, 2) \cup (2, \infty) \longrightarrow \mathbb{R}$

label all x-intercepts, holes, and vertical asymptotes

3. Graph completely using positive-negative analysis the rational function

$$f_3(x) = \frac{x^4 + 4x^3 + 7x^2 + 16x + 12}{x}$$

where $f_3 : (-\infty, 0) \cup (0, \infty) \longrightarrow \mathbb{R}$

label all x-intercepts, holes, and vertical asymptotes

4. Graph completely using positive-negative analysis

the rational function $f_4(x) = \dfrac{x^3 + 8x^2 + 19x + 12}{x^3 - 8}$

where $f_4 : (-\infty, 2) \cup (2, \infty) \longrightarrow \mathbb{R}$

label all x-intercepts, holes, and vertical asymptotes

5. Graph completely using positive-negative analysis

the rational function $f_5(x) = \dfrac{x^2 + 5x^2 - 9x - 45}{x - 3}$

where $f_5 : (-\infty, 3) \cup (3, \infty) \longrightarrow \mathbb{R}$

label all x-intercepts, holes, and vertical asymptotes

6. Graph completely using positive-negative analysis

the rational function $f_6(x) = \dfrac{x - 1}{x^2 - 2x - 35}$

where $f_6 : (-\infty, -5) \cup (-5, 7) \cup (7, \infty) \longrightarrow \mathbb{R}$

label all x-intercepts, holes, and vertical asymptotes

End, last revised 19 October 2004.