Factoring Examples – Trinomials with Leading Coefficients

Example 1: \(3x^2 - 22x + 7\)

Solution Method #1

To factor using the method not in the text, list the factors of the leading coefficient and constant (3 and 7, respectively) to determine which combination of factors in which order gives 22, which is the absolute value of the coefficient of the middle term:

<table>
<thead>
<tr>
<th>Factors of 3</th>
<th>Factors of 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 3</td>
<td>1, 7</td>
</tr>
</tbody>
</table>

Now that all possible pairs of factors have been listed for both 3 and 7, use trial and error to determine which pair of factors of each is paired in which order to obtain a combined value of 22. (In this particular case, the 1 and 3 must be paired with the 1 and 7 since there are no other choices of pairs of factors for either 3 or 7.) Note that since the sign of 7 is positive, the products are added to obtain 22:

\[
\begin{align*}
1 \times 1 &= 1 \\
3 \times 7 &= 21 \\
1 \times 7 &= 7 \\
3 \times 1 &= 3
\end{align*}
\]

\[
\begin{align*}
&\Rightarrow \text{ Add to get } 22 \\
&\Rightarrow \text{ Add to get } 10
\end{align*}
\]

The results above show that 1 must be multiplied by 1, and 3 must be multiplied by 7 to obtain the desired result of negative 22. However, the sign that goes with each number to produce negative 22 must be determined. Since the sum of the products must be negative, both the 1 and 7 that come from the list of factors of 7 must be negative (recall that factoring begins by making sure the leading coefficient is positive so its factors are always both positive). Hence, the final factorization is:

\[
3x^2 - 22x + 7 = (x - 7)(3x - 1)
\]

To check the answer, simply use the FOIL method to work out the product of the binomials obtained in the factorization and verify that the result is, indeed, the original trinomial:

\[
(x - 7)(3x - 1) = 3x^2 - x - 21x + 7 = 3x^2 - 22x + 7
\]

Combining like terms (the results of computing the outer (O) and inner (I) products in the FOIL method) gives the original trinomial. Had the signs been reversed, the result would have been:

\[
(x + 7)(3x + 1) = 3x^2 + x + 21x + 7 = 3x^2 + 22x + 7
\]

The 22 would have been positive instead of negative, which makes the factorization incorrect.
Solution Method #2

The method of factorization in the text is a more algorithmic approach to factoring trinomials with leading coefficients, but it can consume more time and effort than the preceding method. However, it is worth presenting as a reasonable alternative.

To begin factoring using this alternative method, list all signed factors of the product of the leading coefficient and constant (in this case, that product is $3 \times 7 = 21$). Then, compute the sum of each pair of signed factors and select the pair producing the sum equal to the middle coefficient (in this case, look for a sum of negative 22).

<table>
<thead>
<tr>
<th>Signed Factors of 21</th>
<th>Sum of Signed Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 21</td>
<td>22</td>
</tr>
<tr>
<td>−1, −21</td>
<td>−22</td>
</tr>
<tr>
<td>3, 7</td>
<td>10</td>
</tr>
<tr>
<td>−3, −7</td>
<td>−10</td>
</tr>
</tbody>
</table>

Note the desired pair of signed factors is $-1$ and $-21$. Use this pair to rewrite the middle term of the given trinomial as the sum of two terms, the sum of which is the given middle term:

$$3x^2 - 22x + 7 = 3x^2 - x - 21x + 7$$

Then use the Factor by Grouping technique to complete the factorization process:

$$3x^2 - x - 21x + 7 = x(3x - 1) - 7(3x - 1) = (x - 7)(3x - 1)$$

Hence, the final factorization is the following:

$$3x^2 - 22x + 7 = (x - 7)(3x - 1)$$

Both solution methods produce the same final answer, so it is irrelevant which method is used for a given exercise. The first method requires more ingenuity but less physical work, especially if the basic arithmetic is performed mentally; the second method is more procedural but may require significantly more time and physical effort to complete the factorization process. Since both techniques are valid, it is up to each individual to select the method that best suits him or her.
Example 2: \(5x^2 + 3x - 2\)

Solution Method #1

To factor using the method not in the text, list the factors of the leading coefficient and constant (5 and 2, respectively) to determine which combination of factors in which order gives 3, which is the absolute value of the coefficient of the middle term:

<table>
<thead>
<tr>
<th>Factors of 5</th>
<th>Factors of 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 5</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

Now that all possible pairs of factors have been listed for both 5 and 2, use trial and error to determine which pair of factors of each is paired in which order to obtain 3. (In this particular case, the 1 and 5 must be paired with the 1 and 2 since there are no other choices of pairs of factors for either 5 or 2.) Note that since the sign of 2 is negative, the products are subtracted to obtain 3:

\[
\begin{align*}
\{ & 1 \times 1 = 1 \\
& 5 \times 2 = 10 \\
\} \quad \Rightarrow \quad \text{Subtract to get 9} \\
\{ & 1 \times 2 = 2 \\
& 5 \times 1 = 5 \\
\} \quad \Rightarrow \quad \text{Subtract to get 3}
\end{align*}
\]

The results above show that 1 must be multiplied by 2, and 5 must be multiplied by 1 to obtain the desired result of positive 3. However, the sign that goes with each number to produce positive 3 must be determined. Since the larger product must be positive, and the larger product comes from multiplying 5 by 1, the 1 must be positive (recall that factoring begins by making sure the leading coefficient is positive so its factors are always both positive). The larger number does NOT always have the desired sign. Hence, the final factorization is:

\[5x^2 + 3x - 2 = (x + 1)(5x - 2)\]

To check the answer, simply use the FOIL method to work out the product of the binomials obtained in the factorization and verify that the result is, indeed, the original trinomial:

\[(x + 1)(5x - 2) = 5x^2 - 2x + 5x - 2 = 5x^2 + 3x - 2\]

Combining like terms (the results of computing the outer (O) and inner (I) products in the FOIL method) gives the original trinomial. Had the signs been reversed, the result would have been:

\[(x - 1)(5x + 2) = 5x^2 + 2x - 5x - 2 = 5x^2 - 3x - 2\]

The 3 would have been negative instead of positive, which makes the factorization incorrect.
Solution Method #2

The method of factorization in the text is a more algorithmic approach to factoring trinomials with leading coefficients, but it can consume more time and effort than the preceding method. However, it is worth presenting as a reasonable alternative.

To begin factoring using this alternative method, list all signed factors of the product of the leading coefficient and constant (in this case, that product is $5 \times (-2) = -10$). Then, compute the sum of each pair of signed factors and select the pair producing the sum equal to the middle coefficient (in this case, look for a sum of positive 3).

<table>
<thead>
<tr>
<th>Signed Factors of $-10$</th>
<th>Sum of Signed Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1, -10$</td>
<td>$-9$</td>
</tr>
<tr>
<td>$-1, 10$</td>
<td>$9$</td>
</tr>
<tr>
<td>$2, -5$</td>
<td>$-3$</td>
</tr>
<tr>
<td>$-2, 5$</td>
<td>$3$</td>
</tr>
</tbody>
</table>

Note the desired pair of signed factors is $-2$ and $5$. Use this pair to rewrite the middle term of the given trinomial as the sum of two terms, the sum of which is the given middle term:

$$5x^2 + 3x - 2 = 5x^2 - 2x + 5x - 2$$

Then use the Factor by Grouping technique to complete the factorization process:

$$5x^2 - 2x + 5x - 2 = x(5x - 2) + 1(5x - 2) = (x + 1)(5x - 2)$$

Hence, the final factorization is the following:

$$5x^2 + 3x - 2 = (x + 1)(5x - 2)$$

Both solution methods produce the same final answer, so it is irrelevant which method is used for a given exercise. The first method requires more ingenuity but less physical work, especially if the basic arithmetic is performed mentally; the second method is more procedural but may require significantly more time and physical effort to complete the factorization process. Since both techniques are valid, it is up to each individual to select the method that best suits him or her.
Example 3: \(11x^2 + 25x + 6\)

Solution Method #1

To factor using the method not in the text, list the factors of the leading coefficient and constant (6 and 11, respectively) to determine which combination of factors in which order gives 25, which is the absolute value of the coefficient of the middle term:

<table>
<thead>
<tr>
<th>Factors of 11</th>
<th>Factors of 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 11</td>
<td>1, 6</td>
</tr>
<tr>
<td></td>
<td>2, 3</td>
</tr>
</tbody>
</table>

Now that all possible pairs of factors have been listed for both 6 and 11, use trial and error to determine which pair of factors of each is paired in which order to obtain a combined value of 25. Note that since the sign of 11 is positive, the products are added to obtain 25:

\[
\begin{align*}
\{ & \ 1 \times 1 = 1 \quad \Rightarrow \quad \text{Add to get 67} \\
11 \times 6 = 66 & \quad \Rightarrow \quad \text{Add to get 17} \\
\{ & \ 1 \times 1 = 11 \\
11 \times 2 = 2 & \quad \Rightarrow \quad \text{Add to get 35} \\
1 \times 3 = 3 & \quad \Rightarrow \quad \text{Add to get 25} \\
11 \times 2 = 22 & \\
\end{align*}
\]

The results above show that 1 must be multiplied by 3, and 11 must be multiplied by 2 to obtain the desired result of positive 25. However, the sign that goes with each number to produce positive 25 must be determined. Since the sum of the products must be positive, both the 3 and 2 that come from the list of factors of 6 must be positive (recall that factoring begins by making sure the leading coefficient is positive so its factors are always both positive). Hence, the final factorization is:

\[11x^2 + 25x + 6 = (x + 2)(11x + 3)\]

To check the answer, simply use the FOIL method to work out the product of the binomials obtained in the factorization and verify that the result is, indeed, the original trinomial:

\[(x + 2)(11x + 3) = 11x^2 + 3x + 22x + 6 = 11x^2 + 25x + 6\]

Combining like terms (the results of computing the outer (O) and inner (I) products in the FOIL method) gives the original trinomial. Had the signs been reversed, the result would have been:

\[(x - 2)(11x - 3) = 11x^2 - 3x - 22x + 6 = 11x^2 - 25x + 6\]

The 25 would have been negative instead of positive, which makes the factorization incorrect.
Solution Method #2

The method of factorization in the text is a more algorithmic approach to factoring trinomials with leading coefficients, but it can consume more time and effort than the preceding method. However, it is worth presenting as a reasonable alternative.

To begin factoring using this alternative method, list all signed factors of the product of the leading coefficient and constant (in this case, that product is $11 \times 6 = 66$). Then, compute the sum of each pair of signed factors and select the pair producing the sum equal to the middle coefficient (in this case, look for a sum of positive 25).

<table>
<thead>
<tr>
<th>Signed Factors of 66</th>
<th>Sum of Signed Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 66</td>
<td>67</td>
</tr>
<tr>
<td>-1, -66</td>
<td>-67</td>
</tr>
<tr>
<td>2, 33</td>
<td>35</td>
</tr>
<tr>
<td>-2, -33</td>
<td>-35</td>
</tr>
<tr>
<td>3, 22</td>
<td>25</td>
</tr>
<tr>
<td>-3, -22</td>
<td>-25</td>
</tr>
<tr>
<td>6, 11</td>
<td>17</td>
</tr>
<tr>
<td>-6, -11</td>
<td>-17</td>
</tr>
</tbody>
</table>

Note the desired pair of signed factors is 3 and 22. Use this pair to rewrite the middle term of the given trinomial as the sum of two terms, the sum of which is the given middle term:

$$11x^2 + 25x + 6 = 11x^2 + 3x + 22x + 6$$

Then use the Factor by Grouping technique to complete the factorization process:

$$11x^2 + 3x + 22x + 6 = x(11x + 3) + 2(11x + 3) = (x + 2)(11x + 3)$$

Hence, the final factorization is the following:

$$11x^2 + 25x + 6 = (x + 2)(11x + 3)$$

Both solution methods produce the same final answer, so it is irrelevant which method is used for a given exercise. The first method requires more ingenuity but less physical work, especially if the basic arithmetic is performed mentally; the second method is more procedural but may require significantly more time and physical effort to complete the factorization process. Since both techniques are valid, it is up to each individual to select the method that best suits him or her.
Example 4: \(14x^2 - 11x - 15\)

Solution Method #1

To factor using the method not in the text, list the factors of the leading coefficient and constant (14 and 15, respectively) to determine which combination of factors in which order gives 11, which is the absolute value of the coefficient of the middle term:

<table>
<thead>
<tr>
<th>Factors of 14</th>
<th>Factors of 15</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 14</td>
<td>1, 15</td>
</tr>
<tr>
<td>2, 7</td>
<td>3, 5</td>
</tr>
</tbody>
</table>

Now that all possible pairs of factors have been listed for both 14 and 15, use trial and error to determine which pair of factors of each is paired in which order to obtain a combined value of 11. Note that since the sign of 11 is negative, the products are subtracted to obtain 11:

\[
\begin{align*}
\{ 1 \times 1 &= 1 \\
14 \times 15 &= 210 \} & \implies \text{Subtract to get 209} \\
\{ 1 \times 15 &= 15 \\
14 \times 1 &= 14 \} & \implies \text{Subtract to get 1} \\
\{ 1 \times 3 &= 3 \\
14 \times 5 &= 70 \} & \implies \text{Subtract to get 67} \\
\{ 1 \times 5 &= 5 \\
14 \times 3 &= 42 \} & \implies \text{Subtract to get 37} \\
\{ 2 \times 1 &= 2 \\
7 \times 15 &= 105 \} & \implies \text{Subtract to get 103} \\
\{ 2 \times 15 &= 30 \\
7 \times 1 &= 7 \} & \implies \text{Subtract to get 23} \\
\{ 2 \times 3 &= 6 \\
7 \times 5 &= 35 \} & \implies \text{Subtract to get 29} \\
\{ 2 \times 5 &= 10 \\
7 \times 3 &= 21 \} & \implies \text{Subtract to get 11} \\
\end{align*}
\]

The results above show that 2 must be multiplied by 5, and 7 must be multiplied by 3 to obtain the desired result of 11. However, the sign that goes with each number to produce \(-11\) must be determined. Since the larger product must be negative, and the larger product comes from multiplying 7 by 3, the 3 must be negative (recall that factoring always begins by making sure the leading coefficient is positive so its factors are always both positive). The larger number does NOT always have the desired sign. Hence, the final factorization is:

\[14x^2 - 11x - 15 = (2x - 3)(7x + 5)\]
To check the answer, simply use the FOIL method to work out the product of the binomials obtained in the factorization and verify that the result is, indeed, the original trinomial:

\[(2x - 3)(7x + 5) = 14x^2 + 10x - 21x - 15 = 14x^2 - 11x - 15\]

Combining like terms (the results of computing the outer (O) and inner (I) products in the FOIL method) gives the original trinomial. Had the signs been reversed, the result would have been:

\[(2x + 3)(7x - 5) = 14x^2 - 10x + 21x - 15 = 14x^2 + 11x - 15\]

The 11 would have been positive instead of negative, which makes the factorization incorrect.
Solution Method #2

The method of factorization in the text is a more algorithmic approach to factoring trinomials with leading coefficients, but it can consume more time and effort than the preceding method. However, it is worth presenting as a reasonable alternative.

To begin factoring using this alternative method, list all signed factors of the product of the leading coefficient and constant (in this case, that product is \(14 \times (-15) = -210\)). Then, compute the sum of each pair of signed factors and select the pair producing the sum equal to the middle coefficient (in this case, look for a sum of \(-11\)).

<table>
<thead>
<tr>
<th>Signed Factors of (-210)</th>
<th>Sum of Signed Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 , -210</td>
<td>⇒ -209</td>
</tr>
<tr>
<td>-1 , 210</td>
<td>⇒ 209</td>
</tr>
<tr>
<td>2 , -105</td>
<td>⇒ -103</td>
</tr>
<tr>
<td>-2 , 105</td>
<td>⇒ 103</td>
</tr>
<tr>
<td>3 , -70</td>
<td>⇒ -67</td>
</tr>
<tr>
<td>-3 , 70</td>
<td>⇒ 67</td>
</tr>
<tr>
<td>5 , -42</td>
<td>⇒ -37</td>
</tr>
<tr>
<td>-5 , 42</td>
<td>⇒ 37</td>
</tr>
<tr>
<td>6 , -35</td>
<td>⇒ -29</td>
</tr>
<tr>
<td>-6 , 35</td>
<td>⇒ 29</td>
</tr>
<tr>
<td>7 , -30</td>
<td>⇒ -23</td>
</tr>
<tr>
<td>-7 , 30</td>
<td>⇒ 23</td>
</tr>
<tr>
<td>10 , -21</td>
<td>⇒ -11</td>
</tr>
<tr>
<td>-10 , 21</td>
<td>⇒ 11</td>
</tr>
<tr>
<td>14 , -15</td>
<td>⇒ -1</td>
</tr>
<tr>
<td>-14 , 15</td>
<td>⇒ 1</td>
</tr>
</tbody>
</table>

Note the desired pair of signed factors is 10 and -21. Use this pair to rewrite the middle term of the given trinomial as the sum of two terms, the sum of which is the given middle term:

\[14x^2 - 11x - 15 = 14x^2 + 10x - 21x - 15\]

Then use the Factor by Grouping technique to complete the factorization process:

\[14x^2 + 10x - 21x - 15 = 2x(7x + 5) - 3(7x + 5) = (2x - 3)(7x + 5)\]

Hence, the final factorization is the following:

\[14x^2 - 11x - 15 = (2x - 3)(7x + 5)\]

Both solution methods produce the same final answer, so it is irrelevant which method is used for a given exercise. The first method requires more ingenuity but less physical work, especially if the basic arithmetic is performed mentally; the second method is more procedural but may require significantly more time and physical effort to complete the factorization process. Since both techniques are valid, it is up to each individual to select the method that best suits him or her.
Example 5: \[33x^2 + 19x - 10\]

Solution Method #1

To factor using the method not in the text, list the factors of the leading coefficient and constant (33 and 10, respectively) to determine which combination of factors in which order gives 19, which is the absolute value of the coefficient of the middle term:

<table>
<thead>
<tr>
<th>Factors of 33</th>
<th>Factors of 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 33</td>
<td>1, 10</td>
</tr>
<tr>
<td>3, 11</td>
<td>2, 5</td>
</tr>
</tbody>
</table>

Now that all possible pairs of factors have been listed for both 33 and 10, use trial and error to determine which pair of factors of each is paired in which order to obtain a combined value of 19. Note that since the sign of 10 is negative, the products are subtracted to obtain 19:

\[
\begin{align*}
\{1 \times 1 & = 1 \\ 33 \times 10 & = 330 \} \quad \Rightarrow \quad \text{Subtract to get 329} \\
\{1 \times 10 & = 10 \\ 33 \times 1 & = 33 \} \quad \Rightarrow \quad \text{Subtract to get 23} \\
\{1 \times 2 & = 2 \\ 33 \times 5 & = 165 \} \quad \Rightarrow \quad \text{Subtract to get 163} \\
\{1 \times 5 & = 5 \\ 33 \times 2 & = 66 \} \quad \Rightarrow \quad \text{Subtract to get 61} \\
\{3 \times 1 & = 3 \\ 11 \times 10 & = 110 \} \quad \Rightarrow \quad \text{Subtract to get 107} \\
\{3 \times 10 & = 30 \\ 11 \times 1 & = 11 \} \quad \Rightarrow \quad \text{Subtract to get 19} \\
\{3 \times 2 & = 6 \\ 11 \times 5 & = 55 \} \quad \Rightarrow \quad \text{Subtract to get 49} \\
\{3 \times 5 & = 15 \\ 11 \times 2 & = 22 \} \quad \Rightarrow \quad \text{Subtract to get 7}
\end{align*}
\]

The results above show that 3 must be multiplied by 10 and 11 by 1 to obtain the desired result of positive 19. However, the sign that goes with each number to produce positive 19 must be determined. Since the larger product must be positive, and the larger product comes from multiplying 3 by 10, the 10 must be positive (recall that factoring always begins by making sure the leading coefficient is positive so its factors are always both positive). The larger number does NOT always have the desired sign, although it does in this particular example. Hence, the final factorization is:

\[33x^2 + 19x - 10 = (3x - 1)(11x + 10)\]
To check the answer, simply use the FOIL method to work out the product of the binomials obtained in the factorization and verify that the result is, indeed, the original trinomial:

\[(3x - 1)(11x + 10) = 33x^2 + 30x - 11x - 10 = 33x^2 + 19x - 10\]

Combining like terms (the results of computing the outer (O) and inner (I) products in the FOIL method) gives the original trinomial. Had the signs been reversed, the result would have been:

\[(3x + 1)(11x - 10) = 33x^2 + 11x - 10 = 33x^2 - 19x - 10\]

The 19 would have been negative instead of positive, which makes the factorization incorrect.
Solution Method #2

The method of factorization in the text is a more algorithmic approach to factoring trinomials with leading coefficients, but it can consume more time and effort than the preceding method. However, it is worth presenting as a reasonable alternative.

To begin factoring using this alternative method, list *all signed factors of the product of the leading coefficient and constant* (in this case, that product is $33 \times (-10) = -330$). Then, compute the sum of each pair of signed factors and select the pair producing the sum equal to the middle coefficient (in this case, look for a sum of positive 19).

<table>
<thead>
<tr>
<th>Signed Factors of $-330$</th>
<th>Sum of Signed Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$, $-330$</td>
<td>$-329$</td>
</tr>
<tr>
<td>$-1$, $330$</td>
<td>$329$</td>
</tr>
<tr>
<td>$2$, $-165$</td>
<td>$-163$</td>
</tr>
<tr>
<td>$-2$, $165$</td>
<td>$163$</td>
</tr>
<tr>
<td>$3$, $-110$</td>
<td>$-107$</td>
</tr>
<tr>
<td>$-3$, $110$</td>
<td>$107$</td>
</tr>
<tr>
<td>$5$, $-66$</td>
<td>$-61$</td>
</tr>
<tr>
<td>$-5$, $66$</td>
<td>$61$</td>
</tr>
<tr>
<td>$6$, $-55$</td>
<td>$-49$</td>
</tr>
<tr>
<td>$-6$, $55$</td>
<td>$49$</td>
</tr>
<tr>
<td>$10$, $-33$</td>
<td>$-23$</td>
</tr>
<tr>
<td>$-10$, $33$</td>
<td>$23$</td>
</tr>
<tr>
<td>$11$, $-30$</td>
<td>$-19$</td>
</tr>
<tr>
<td>$-11$, $30$</td>
<td>$19$</td>
</tr>
<tr>
<td>$15$, $-22$</td>
<td>$-7$</td>
</tr>
<tr>
<td>$-15$, $22$</td>
<td>$7$</td>
</tr>
</tbody>
</table>

Note the desired pair of signed factors is $-11$ and $30$. Use this pair to rewrite the middle term of the given trinomial as the sum of two terms, the sum of which is the given middle term:

$$33x^2 + 19x - 10 = 33x^2 - 11x + 30x - 10$$

Then use the Factor by Grouping technique to complete the factorization process:

$$33x^2 - 11x + 30x - 10 = 11x(3x - 1) + 10(3x - 1)$$

Hence, the final factorization is the following:

$$33x^2 + 19x - 10 = (11x + 10)(3x - 1)$$

Both solution methods produce the same final answer, so it is irrelevant which method is used for a given exercise. The first method requires more ingenuity but less physical work, especially if the basic arithmetic is performed mentally; the second method is more procedural but may require significantly more time and physical effort to complete the factorization process. Since both techniques are valid, it is up to each individual to select the method that best suits him or her.