Meanings of Basic Arithmetic Operations in Mathematics

Addition: Generally indicated by the word "OR" 
"PLUS" or "SUM" or "TOTAL" or "TOGETHER" 
Used to put complete pieces together in complex situations 
Useful in counting possibilities involving choices among unique options

Subtraction: "MINUS" or "DIFFERENCE" 
Used to remove complete pieces from complex situations 
Useful in working with dependent (i.e., overlapping) events

Multiplication: Generally indicated by the word "AND" 
"TIMES" or "PRODUCT" 
Used to combine individual characteristics to form one complete piece 
Useful in computing probabilities involving sequences of events

Division: "DIVIDED BY" or "QUOTIENT" 
Used to eliminate individual characteristics from one complete piece 
Useful in interpreting Permutations and Combinations

Factorials

For any integer \( n > 0 \), \( n! = n(n-1)(n-2)\cdots3 \cdot 2 \cdot 1 \); by definition, \( 0! = 1 \).

Permutations

For any integers \( n \) and \( r \) with \( n > 0 \) and \( 0 \leq r \leq n \), \( P(n, r) = nP_r = \frac{n!}{(n-r)!} \).

Combinations

For any integers \( n \) and \( r \) with \( n > 0 \) and \( 0 \leq r \leq n \), \( C(n, r) = nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!} \).

Dependent vs. Independent Events

Dependent: Outcome of one event has a measurable effect on another 
Independent: Outcomes of all events have no effect on any others

Binary [Probability] (Also Referred to as Bernoulli Experiments)

Situations involving exactly the same two options at each stage of computation 
Many complex situations can be interpreted as binary (i.e., Bernoulli) experiments
Counting Techniques

Tree Diagrams

All possible outcomes are visually represented by their own branches.

Example:

List all possible ways to form a 3-digit number from the digits 0, 1, and 2 if the first digit cannot be 0, and no two consecutive digits may be even.

```
0          1          101
    0          110
    1          111
    2          112

1          1
    1          121
    2

2          1
    0          210
    1          211
    2
```

Solution: \{ 101, 110, 111, 112, 121, 210, 211, 212 \}

Product Rule

Multiply the number of possibilities for each part of an event to obtain a total.

Examples:

1. How many complete dinners can be created from a menu with 5 appetizers, 8 entrées, and 4 desserts if a complete dinner consists of one appetizer, one entrée, and one dessert?

   \[ 5 \cdot 8 \cdot 4 = 160 \]

2. How many unique license plates exist if each license plate must consist of 3 letters followed by 4 digits?

   \[ 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175,760,000 \]
Sum Rule

Add the number of possibilities for different ways to complete an event to obtain a total.

Examples:

1. How many ways can a postsecondary student be selected from a group of 120 undergraduate students and 56 graduate students?

   \[120 + 56 = 176\]

2. How many unique license plates exist if each license plate must consist of 3 letters followed by 4 digits or 3 letters followed by 3 digits?

   - 3 letters followed by 4 digits: \(26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175,760,000\)
   - 3 letters followed by 3 digits: \(26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000\)

   Total: \(175,760,000 + 17,576,000 = 193,336,000\)

Subtraction Rule (Inclusion-Exclusion Principle)

Items cannot be counted more than once when applying the Sum Rule; subtract overlapping quantities to compensate, but be sure to include all possibilities exactly once.

Examples:

1. How many students are in a group of students consisting of 135 computer science majors, 112 mathematics majors, and 43 computer science/mathematics double majors?

   \[135 + 112 - 43 = 204\]

2. How many 3-letter "words" (sequences of letters; not necessarily real words) begin or end with a vowel?

   - Begin with a vowel: \(5 \cdot 26 \cdot 26 = 3,380\)
   - End with a vowel: \(26 \cdot 26 \cdot 5 = 3,380\)
   - Begin and end with a vowel: \(5 \cdot 26 \cdot 5 = 650\)

   Total: \(3,380 + 3,380 - 650 = 6,110\)
Fill-in-the-Blanks Technique

Draw a line to represent a slot for each location an object must be placed, write the number of possible options that can go in each slot, and multiply the results to obtain a total number of possible outcomes. For more complicated situations, fill in the slots with their respective values by progressing from the slot with the most restrictions (or the least options) to the slot with the least restrictions (or the most options), not necessarily left to right. If necessary, break the exercise into multiple cases, calculate each case individually, and add the results together to obtain a final answer.

Examples:

1. How many ways can a person roll a die, then flip a coin, then spin a 1-to-10 spinner?

   \[ 6 \cdot 2 \cdot 10 = 120 \]

2. How many even 4-digit numbers greater than 7499 with no repeated digits exist?

   - **7500s and 7900s:**
     \[
     \frac{1}{7} \cdot \frac{2}{5,9} \cdot \frac{7}{0 - 9} \cdot \frac{5}{0,2,4,6,8} = 70
     \]

   - **7600s and 7800s:**
     \[
     \frac{1}{7} \cdot \frac{2}{6,8} \cdot \frac{7}{0 - 9} \cdot \frac{3}{0,2,4} + \frac{1}{7} \cdot \frac{2}{6,8} \cdot \frac{7}{0 - 9} \cdot \frac{1}{6,8} = 56
     \]

   - **8000s:**
     \[
     \frac{1}{8} \cdot \frac{8}{0 - 9} \cdot \frac{7}{0 - 9} \cdot \frac{4}{0,2,4,6} = 224
     \]

   - **9000s:**
     \[
     \frac{1}{9} \cdot \frac{8}{0 - 9} \cdot \frac{7}{0 - 9} \cdot \frac{5}{0,2,4,6,8} = 280
     \]

   Total: \[ 70 + 56 + 224 + 280 = 630 \]
Exponentiation (Permutations with Repetition/Replacement)

The number of ways to arrange $r$ objects from a set of $n$ objects if any object can be repeated is $n^r$.

**Examples:**

1. *How many 8-digit bit strings exist (a bit string is a sequence of binary digits, each of which is a 0 or 1; an example of an 8-digit bit string is 10010110)?*
   
   Either 0 or 1 at each position: $2^8 = 256$

2. *How many ways can a student randomly guess the answers to a 10-question multiple choice quiz if each question has possible answers of (a), (b), (c), and (d)?*
   
   $4^{10} = 1,048,576$

Factorials

There are $n!$ ways to arrange a group of $n$ objects if the order in which they are arranged matters.

**Examples:**

1. *How many ways can a group of 10 students line up to buy ice cream?*
   
   $10! = 3,628,800$

2. *How many ways can 5 textbooks be arranged on a shelf?*
   
   $5! = 120$

3. *How many ways can 5 elementary school students, 6 high school students, and 7 college students be lined up such that all students at the same level of school are grouped next to one another?*
   
   $(5! \cdot 6! \cdot 7! \cdot 3! = 2,612,736,000$

4. *How many ways can the letters in the set { A, B, C, D, E, F, G } be arranged such that the string "BAD" must occur in the outcome?*

   Consider "BAD" as a single object and the letters C, E, F, and G separately: $5! = 120$
Permutations

There are $P(n, r)$ ways to arrange $r$ of a group of $n$ objects if the order they are arranged matters.

Example:

How many ways can 3 of a group of 10 students line up to buy ice cream?

$$P(10, 3) = \frac{10!}{7!} = 720$$

Combinations

There are $C(n, r)$ ways to arrange $r$ of a group of $n$ objects if the order in which they are arranged is irrelevant.

Examples:

1. How many 5-card poker hands can be dealt from a standard deck of playing cards?

$$C(52, 5) = \frac{52!}{47!5!} = 2,598,960$$

2. How many 3-member committees can be selected from a group of 12 people?

$$C(12, 3) = \frac{12!}{9!3!} = 220$$

3. How many ways can a 7-member board of trustees be created from a group of 10 men and 8 women if 4 members must be male and 3 members must be female?

$$C(10, 4) \cdot C(8, 3) = \frac{10!}{6!4!} \cdot \frac{8!}{5!3!} = 210 \cdot 56 = 11,760$$
Pascal's Triangle

This is a visual computational technique for determining values of combinations with applications including determining coefficients in binomial expansions. Each row begins and ends with 1 (with a special Row #0 at the top consisting of a single 1) and is written such that consecutive rows are aligned using alternating staggers with each row containing one value more than the preceding row. Aside from the leading and trailing 1 in each row, all other values are found by adding the values staggered to either side of the desired location in any row. The first several rows are as follows:

Row 0: 1
Row 1: 1 1
Row 2: 1 2 1
Row 3: 1 3 3 1
Row 4: 1 4 6 4 1
Row 5: 1 5 10 10 5 1
Row 6: 1 6 15 20 15 6 1
Row 7: 1 7 21 35 35 21 7 1
Row 8: 1 8 28 56 70 56 28 8 1
Row 9: 1 9 36 84 126 126 84 36 9 1
Row 10: 1 10 45 120 210 252 210 120 45 10 1

To illustrate its use, consider the computation of \( C(8, 3) \). If we number the position of each value in a given row such that the first position is 0 and the last position is the same as the row number, we look at the value at position 3 in Row 8 to determine the value of \( C(8, 3) \). That value is 56. If we calculate the value of \( C(8, 3) \) using the definition of a combination, we obtain the same value:

\[
C(8, 3) = \frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \frac{336}{6} = 56
\]
Permutations with Indistinguishable Objects

The number of ways to arrange a set of \( n \) objects consisting \( n_1 \) identical objects of type 1, \( n_2 \) identical objects of type 2, and so on up to \( n_k \) identical objects of type \( k \), is \( \frac{n!}{n_1!n_2!\cdots n_k!} \).

Example:

*How many unique ways can one arrange the letters of the word MISSISSIPPI?*

\[
\frac{11!}{1!4!4!2!} = \frac{39,916,800}{1 \cdot 24 \cdot 24 \cdot 2} = 34,650
\]

Pigeonhole Principle (Dirichlet Drawer Principle)

If the number of objects being distributed into a set of containers is greater than the number of containers, some container must contain more than one object.

Examples:

1. *How many students must take a test to guarantee at least two of them receive the same letter grade (A, B, C, D, or F; no plus or minus)?*

   6

2. *How many times must a die be rolled to guarantee a number occurs twice?*

   7

Generalized Pigeonhole Principle

If \( n \) objects are distributed into \( k \) containers, at least one container holds \( \frac{n}{k} \) objects, rounded up.

Examples:

1. *How many people can be guaranteed to share a birthday (assume 366 possible birthdays to include Leap Day, February 29) in a group of 10,000 people?*

   \[
   \frac{10,000}{366} \approx 27.3 \quad \Rightarrow \quad \text{at least 28 people must share a birthday}
   \]

2. *How many times must a pair of dice be thrown to guarantee some total shown on the dice occurs at least 5 times?*

   11 possible totals on each throw: \{ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \}

   “At least 5” should be interpreted as “greater than 4”

   \[
   \frac{n}{11} > 4 \quad \Rightarrow \quad n > 44 \quad \Rightarrow \quad n = 45
   \]
Combinations with Repetition/Replacement

The number of ways to select \( r \) objects from a set containing \( n \) types of objects such that the type of object selected can be repeated and each of the \( n \) types is available for each selection is:

\[
C(n + r - 1, r) = C(n + r - 1, n - 1)
\]

Examples:

1. **How many ways can 7 coins be selected from a pile containing pennies, nickels, dimes, and quarters if the order in which the coins are selected is irrelevant and the pile contains at least 7 of each type of coin?**

\[
C(4 + 7 - 1, 7) = C(10, 7) = 120
\]

2. **How many solutions exist to the equation \( x_1 + x_2 + x_3 + x_4 = 15 \) such that the value of each variable is a nonnegative integer?**

This can be thought of as the number of ways to select 15 items from a set consisting of 4 elements such that each element is replaced after it is selected; alternatively, consider the situation in which 4 unique objects are in a bag, and 15 random selections are made, recording which type is drawn and replacing it each time:

\[
C(4 + 15 - 1, 15) = C(18, 15) = 816
\]

Distributing Distinguishable Objects into Distinguishable Boxes

The number of ways to distribute \( n \) distinguishable objects into \( k \) distinguishable boxes such that \( n_i \) objects are placed in box \( i \) is \( \frac{n!}{n_1!n_2!\cdots n_k!} \).

Example:

**How many ways can a 5-card poker hand be dealt to each of 4 players from a standard deck?**

52 cards to be distributed to 4 players leaves 32 cards left to consider along with the 20 dealt:

\[
\frac{52!}{5!5!5!5!32!} = 1,478,262,843,475,644,020,034,240
\]
Distributing Indistinguishable Objects into Distinguishable Boxes

This situation is identical to computing the number of combinations with repetition:

\[ C(n + r - 1, r) = C(n + r - 1, n - 1) \]

Example:

How many ways can 12 identical blocks be placed into 9 numbered boxes?

\[ C(9 + 12 - 1, 12) = C(20, 12) = 125,970 \]

Distributing Distinguishable Objects into Indistinguishable Boxes

No simple closed formula exists for this situation, although a complicated formula involving summations does exist. It is usually easiest to write out all possibilities.

Distributing Indistinguishable Objects into Indistinguishable Boxes

No simple closed formula exists for this situation. It is usually easiest to write out all possibilities.
Probability Concepts

Discrete Probability

If the sample space (i.e., the set of all possible outcomes), \( S \), for a given experiment and the set of desired outcomes, \( E \), are both countable, the probability that \( E \) occurs is given by:

\[
P(E) = \frac{n(E)}{n(S)}
\]

In sum, the counting techniques previously described in this packet can be applied to the sample space, \( S \), and the event of interest, \( E \), to obtain their respective sizes, and the probability that the event, \( E \), occurs is obtained by dividing their values.

The probability of any event occurring is always between 0 and 1, where any event with a probability of 0 is an impossibility, and any event with a probability of 1 is a certainty.

Examples:

1. What is the probability that an individual wins a lottery in which he or she must correctly pick all 6 numbers randomly selected from 1 through 60?

   Number of ways to win: 1
   Total number of outcomes: \( \binom{60}{6} = 50,063,860 \)
   Probability of winning: \( \frac{1}{50,063,860} \approx 0.0000001997 \approx 0.000002\% \)

2. Find the probability that a 5-card poker hand is a full house.

   Note that a full house consists of 3 of one denomination and 2 of another, and that there are 13 denominations to start.

   Number of full houses: \( P(13, 2) \cdot \binom{4}{3} \cdot \binom{4}{2} = 78 \cdot 4 \cdot 6 = 3,744 \)
   Total number of 5-card poker hands: \( \binom{52}{5} = 2,598,960 \)
   Probability of dealing a full house: \( \frac{3,744}{2,598,960} \approx 0.0014 = 0.14\% \)

3. What is the probability that a student gets at least 8 out of 10 questions correct on a quiz consisting of 10 True/False questions if he or she randomly guesses all the answers?

   Number of ways to get 8 correct: \( \binom{10}{8} = 45 \)
   Number of ways to get 9 correct: \( \binom{10}{9} = 10 \)
   Number of ways to get 10 correct: \( \binom{10}{10} = 1 \)
   Total number of ways to answer all questions: \( 2^{10} = 1,024 \)
   Probability of at least 8 correct by guessing: \( \frac{45 + 10 + 1}{1,024} = \frac{56}{1,024} = \frac{7}{128} \approx 0.0547 = 5.47\% \)
Complement Rule

The probability that a given event, $E$, occurs is given by $P(E) = 1 - P(\overline{E})$, where $P(\overline{E})$ is the probability that event $E$ does not occur, or the probability that the complement of $E$ occurs.

Example:

Find the probability that a student gets at least 2 out of 10 questions wrong on a quiz consisting of 10 multiple choice questions with 4 answers each if the student randomly guesses the answers.

\[
P(\text{at least 2 wrong}) = 1 - P(\text{no more than 1 wrong})
\]
Number of ways to get 0 wrong: $C(10, 0) = 1$
Number of ways to get 1 wrong: $C(10, 1) = 10$
Total number of ways to answer all questions: $4^{10} = 1,048,576$
Probability of no more than 1 wrong: $\frac{1 + 10}{1,048,576} = \frac{11}{1,048,576} \approx 0.00001049 \approx 0.001\%$
Probability of at least 2 wrong: $1 - 0.00001049 = 0.99998951 \approx 99.999\%$

Binomial Probability (Bernoulli Trials)

If a probability experiment can be expressed as a repetition of identical independent trials of an experiment such that the probability of success for each trial is $p$ and the probability of failure for each trial is $q = 1 - p$, then the probability of obtaining exactly $k$ successes in $n$ trials is given by $C(n, k)p^kq^{n-k}$.

Examples:

1. What is the probability of randomly guessing exactly 8 questions correctly on a quiz consisting of 10 multiple choice questions with 3 possible answer for each?

\[
P(8) = C(10, 8)\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^2 = (45)\left(\frac{1}{6,561}\right)\left(\frac{4}{9}\right) \approx 0.003048 \approx 0.3\%\]

2. Find the probability that at least 11 eggs in a carton of one dozen are intact if the probability of a broken egg is 1.5%.

\[
P(11) = C(12, 11)(0.985)^{11}(0.015)^1 \approx (12)(0.8468)(0.015) \approx 0.1524 = 15.24\%
\]
\[
P(12) = C(12, 12)(0.985)^{12}(0.015)^0 \approx (1)(0.8341)(1) = 0.8341 = 83.41\%
\]

Total desired probability: $0.1524 + 0.8341 = 0.9865 = 98.65\%$
Arithmetic Facts Involving Counting Techniques

\[ 0! = 1 \]

\[ P(n, 0) = \frac{n!}{n!} = 1 \]

*Example:* \( P(15, 0) = 1 \)

\[ P(n, 1) = \frac{n!}{(n-1)!} = n \]

*Example:* \( P(47, 1) = 47 \)

\[ P(n, n) = \frac{n!}{0!} = n! \]

*Example:* \( P(8, 8) = 8! = 40,320 \)

\[ C(n, 0) = \frac{n!}{n!0!} = 1 \]

*Example:* \( C(23, 0) = 1 \)

\[ C(n, 1) = \frac{n!}{(n-1)!1!} = n \]

*Example:* \( C(60, 1) = 60 \)

\[ C(n, n-1) = \frac{n!}{1!(n-1)!} = n \]

*Example:* \( C(36, 35) = 36 \)

\[ C(n, n) = \frac{n!}{0!n!} = 1 \]

*Example:* \( C(80, 80) = 1 \)

\[ C(n, r) = C(n, n - r) \]

*Example:*

\[ C(10, 7) = \frac{10!}{3!7!} = 120 \]

\[ C(10, 3) = \frac{10!}{7!3!} = 120 \]